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I.—EXPOSITION OF A GENERAL THEORY OF LINEAR TRANSFORMATIONS. PART I.

By George Boole.

1. The transformation of homogeneous functions by linear substitutions, is an important and oft-recurring problem of analysis. In the *Mécanique Analytique* of Lagrange, it occupies a very prominent place, and it has been made the subject of a special memoir by Laplace. More recently it has engaged the attention of Lebesgue and Jacobi; the former of whom has extended his investigations to homogeneous functions of the second degree, and of an indefinite number of variables, while the latter has applied the results of such inquiries to the transformation of multiple integrals. A memoir on this subject has also been given to the world by Cauchy; and an ingenious paper by Professor De Morgan, on its geometrical relations, will be found in the 5th volume of the *Cambridge Philosophical Transactions*.

The most general conclusion to which the labours of the above-mentioned writers have led, is, that it is always possible to take away the products of the variables x_1, x_2, ... x_m, from a proposed homogeneous function of the second degree, Q, by the linear substitution of a new set of variables, y_1, y_2, ... y_m, connected with the original ones by the relation

$$x_1^2 + x_2^2 \ldots + x_m^2 = y_1^2 + y_2^2 \ldots + y_m^2 \ldots . (1);$$

or in other words, to determine, subject to (1), the values of

the coefficients A_1, A_2, .. A_m, in the equation of transformation,

$$Q = A_1 y_1^2 + A_2 y_2^2 \ldots + A_m y_m^2 \ldots\ldots (2).$$

And the method commonly employed in this investigation has been, to substitute, in place of the variables involved in either member of (2), a series of linear functions of the variables involved in the opposite member, to equate coefficients, and to eliminate the unknown constants by aid of the equations of condition similarly obtained from (1). It is in the effecting of this elimination that the principal difficulty of the problem consists; a difficulty arising from the very principle of the method of solution, and therefore not to be evaded; a difficulty moreover so great, that no one has yet shewn how it is to be overcome, when the degree of the function to be transformed rises above the second.

In the above remarks, it is not however intended to convey the idea that this elimination is impossible. Were the final results of the elimination known, it would at once be seen by what combinations of our equations those results might be produced; but this fact brings us no nearer to their actual discovery. Indeed it must on the slightest consideration be manifest, that no such principle of investigation can suffice to the requirements of a problem, which, alike in its primary analysis and in the forms of its final solution, will be shewn to rest on the doctrine of developments, and to involve the processes of the Differential Calculus.

2. The equations (1) and (2) are evidently particular cases of the homogeneous system,

$$h_2(x_1, x_2, \ldots x_m) = h'_2(y_1, y_2, \ldots y_m) \ldots (A_1),$$
$$H_2(x_1, x_2, \ldots x_m) = H'_2(y_1, y_2, \ldots y_m) \ldots (B_1),$$

in which h_2, h'_2, H_2, H'_2, designate homogeneous functions of the second degree; and these again of the more general system,

$$h_n(x_1, x_2, \ldots x_m) = h'_n(y_1, y_2, \ldots y_m) \ldots (A_2),$$
$$H_n(x_1, x_2, \ldots x_m) = H'_n(y_1, y_2, \ldots y_m) \ldots (B_2),$$

h_n, h_n', &c. indicating, in accordance with the above employed notation, homogeneous functions of the n^{th} degree; and the problem in this case mainly refers to the determination of the mutual relations of the constants in (A_2) and (B_2), on the assumption that the second members of those equations are formed respectively from their first, by the same system of linear substitutions. The equations (A_2) and (B_2), I shall,

throughout the remainder of this paper, represent under the abbreviated forms

$$q = r \,.\, (A_3),$$
$$Q = R \,.\, (B_3).$$

Let ν be the number of the terms in the homogeneous function of the n^{th} degree with m variables, and let ι represent in succession each of the series of integers from $m+1$ to ν inclusive; also, let $\alpha, \beta, \gamma \ldots \mu$ be indefinite integers, varying from 0 to $n-1$ inclusive, and entering into every possible combination subject to the condition of homogeneity,

$$\alpha + \beta + \gamma \ldots + \mu = n;$$

then may the functions $q, r, Q, R,$ be represented under the general forms

$$\left.\begin{aligned} q &= a_1 x_1^n + a_2 x_2^n \ldots + a_m x_m^n + \Sigma a_\iota x_1^\alpha x_2^\beta \ldots x_m^\mu \\ r &= b_1 y_1^n + b_2 y_2^n \ldots + b_m y_m^n + \Sigma b_\iota y_1^\alpha y_2^\beta \ldots y_m^\mu \\ Q &= A_1 x_1^n + A_2 x_2^n \ldots + A_m x_m^n + \Sigma A_\iota x_1^\alpha x_2^\beta \ldots x_m^\mu \\ R &= B_1 y_1^n + B_2 y_2^n \ldots + B_m y_m^n + \Sigma B_\iota y_1^\alpha y_2^\beta \ldots y_m^\mu \end{aligned}\right\} \ldots (3),$$

and our primary object, as above observed, is to determine the relations by which the 4ν quantities $a_1, a_2 \ldots a_\nu, \ldots b_1, b_2 \ldots b_\nu, A_1, A_2 \ldots A_\nu, B_1, B_2 \ldots B_\nu$, are held in mutual dependence.

By Taylor's theorem, it is evident that the equations (A_3), (B_3), cannot exist in their utmost generality, unless the various orders of equations thence obtained by successive differentiation, are also satisfied. Confining our attention to those of the first order, and differentiating relatively to $y_1, y_2 \ldots y_m$, we have, from (B_3),

$$\left.\begin{array}{l} \dfrac{dQ}{dx_1}\dfrac{dx_1}{dy_1} + \dfrac{dQ}{dx_2}\dfrac{dx_2}{dy_1} \ldots + \dfrac{dQ}{dx_m}\dfrac{dx_m}{dy_1} = \dfrac{dR}{dy_1} \\ \dfrac{dQ}{dx_1}\dfrac{dx_1}{dy_2} + \dfrac{dQ}{dx_2}\dfrac{dx_2}{dy_2} \ldots + \dfrac{dQ}{dx_m}\dfrac{dx_m}{dy_2} = \dfrac{dR}{dy_2} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{dQ}{dx_1}\dfrac{dx_1}{dy_m} + \dfrac{dQ}{dx_2}\dfrac{dx_2}{dy_m} \ldots + \dfrac{dQ}{dx_m}\dfrac{dx_m}{dy_m} = \dfrac{dR}{dy_m} \end{array}\right\} \ldots (4).$$

3. Let us suppose the actual linear relations among the two sets of variables to be represented under the forms,

$$\left.\begin{aligned} x_1 &= \lambda_1 y_1 + \lambda_2 y_2 \ldots + \lambda_m y_m \\ x_2 &= \mu_1 y_1 + \mu_2 y_2 \ldots + \mu_m y_m \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_m &= \rho_1 y_1 + \rho_2 y_2 \ldots + \rho_m y_m \end{aligned}\right\} \ldots (5).$$

Then do the above equations (4) become

$$\left.\begin{array}{l}\lambda_1 \dfrac{dQ}{dx_1} + \mu_1 \dfrac{dQ}{dx_2} \ldots + \rho_1 \dfrac{dQ}{dx_m} = \dfrac{dR}{dy_1} \\ \lambda_2 \dfrac{dQ}{dx_1} + \mu_2 \dfrac{dQ}{dx_2} \ldots + \rho_2 \dfrac{dQ}{dx_m} = \dfrac{dR}{dy_2} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \lambda_m \dfrac{dQ}{dx_1} + \mu_m \dfrac{dQ}{dx_2} \ldots + \rho_m \dfrac{dQ}{dx_m} = \dfrac{dR}{dy_m}\end{array}\right\} \ldots (6).$$

We suppose the coefficients $\lambda_1, \lambda_2, \ldots \lambda_m, \mu_1, \mu_2 \ldots \mu_m$, &c. in the linear theorems (5) to be of finite value; and on this supposition it is clear that the assumption in (6) of the simultaneous conditions,

$$\frac{dQ}{dx_1} = 0, \quad \frac{dQ}{dx_2} = 0, \ldots \frac{dQ}{dx_m} = 0 \ldots . (7)$$

will induce, as a necessary consequence, the fulfilment of the simultaneous and similar conditions

$$\frac{dR}{dy_1} = 0, \quad \frac{dR}{dy_2} = 0, \ldots \frac{dR}{dy_m} = 0 \ldots . (8).$$

The converse of this proposition is not so universally true.

If we suppose the second members of (6) to vanish, and linearly eliminate $\dfrac{dQ}{dx_1}, \dfrac{dQ}{dx_2}, \ldots \dfrac{dQ}{dx_m}$, from the first members thus equated to 0, we shall obtain a final equation among the constants,

$$F(\lambda_1, \mu_1, \ldots \rho_1, \ldots \lambda_m, \mu_m, \ldots \rho_m) = 0 \ldots . (9),$$

which, if satisfied, will indicate, that the proposed conditions, (8), may coexist, without the simultaneous evanescence of $\dfrac{dQ}{dx_1}, \dfrac{dQ}{dx_2}, \ldots \dfrac{dQ}{dx_m}$. These circumstances are here noticed, because they will be found to reappear, as the cause of certain peculiarities in the final solution. On the nature and meaning of the condition (9), it will for the present be sufficient to remark, that it analytically corresponds to those cases in which, while definite values are attributed to the one set of variables $y_1, y_2, \ldots y_m$, those of the other set $x_1, x_2, \ldots x_m$, become arbitrary or infinite, and that its geometrical interpretation has reference to certain cases of impossible transformation, such as, for example, the change of co-ordinates from a pair of axes having a given inclination to another pair mutually coinciding. Omitting therefore the further

consideration of this case of failure, we proceed to examine the conclusions which may be drawn from the otherwise dependent, and always compatible systems (7) and (8).

4. Employing the value of Q given in (3), the system (7) becomes

$$\left.\begin{array}{l} nA_1x_1^{n-1} + \Sigma A_\iota a x_1^{a-1}x_2^{\beta} \ldots x_m^{\mu} = 0 \\ nA_2x_2^{n-1} + \Sigma A_\iota \beta x_1^{a} x_2^{\beta-1} \ldots x_m^{\mu} = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ nA_mx_m^{n-1} + \Sigma A_\iota \mu x_1^{a} x_2^{\beta} \ldots x_m^{\mu-1} = 0 \end{array}\right\} \ldots (10).$$

These equations are homogeneous and of the $(n-1)^{\text{th}}$ degree: we are now to inquire under what conditions they may be satisfied. The first which presents itself is found in the supposition, that $x_1, x_2, \ldots x_m$, simultaneously vanish; but this leads to no ulterior result. The only remaining one will be determined by examining the result arising from the elimination of the variables. To this object we therefore now direct our attention.

If to any two equations of the n^{th} degree,

$$ax^n + bx^{n-1} + cx^{n-2} \ldots + k = 0 \ldots . (11),$$

$$a'x^n + b'x^{n-1} + cx'^{n-2} \ldots + k' = 0 \ldots . (12),$$

we apply the process employed in finding a common measure, and equate the successive remainders to 0, which in such case we evidently may do, we shall in the first stage of the process obtain an equation of the $(n-1)^{\text{th}}$ degree, in the next stage we shall have an equation of the $(n-2)^{\text{th}}$ degree; and thus at each successive stage of the process depressing the degree of our resulting equation by unity, shall finally arrive at an equation involving only the coefficients a, b, a', b', &c. with the constants k and k', and which may be regarded as the result of the final elimination of x from (11) and (12).

By applying this method to equations (1) and (2) of (10), we shall be able to eliminate x_1, and repeating the process successively on the independent pairs (2) and (3), (3) and (4), $(m-1)$ and (m), of the same system, shall form a system of $m-1$ equations, involving only the variables $x_2, x_3, \ldots x_m$, and from which x_1 will have entirely disappeared. These equations may be treated in the same way as those of (10), and will lead by the elimination of x_2, to a system of $m-2$ equations involving only the variables, $x_3, x_4, \ldots x_m$. Thus by successive stages shall we obtain from the m equations (10) with m variables,

$m-1$ equations involving $m-1$ variables,
$m-2$ equations . . . . $m-2$ variables,
. . . .
2 equations . . . . 2 variables,
1 final equation . . . . constants;

since all these equations are, like the original ones, homogeneous, and from the last the variable will divide out. It is also to be observed, that the two equations given in the last step but one of the process of reduction, will be linear, as will be more distinctly seen in the examples given below. This circumstance is important, because it restricts the general solution to the condition of linearity among the variables; the reason will appear in the sequel.

The last obtained of the above column of results, we shall designate by the symbol θ, applied to the original function Q. Thus, if, to adopt the common notation, Q were a homogeneous function of the second degree, of the form,

$$Q = Ax^2 + 2Bxy + Cy^2 \ldots (13);$$

then on eliminating from the derived equations,

$$Ax + By = 0,$$
$$Bx + Cy = 0,$$

(for $\dfrac{dQ}{dx} = 2Ax + 2By$, and $\dfrac{dQ}{dy} = 2Bx + 2Cy$), we find

$$\theta(Q) = B^2 - AC, \text{ or } AC - B^2 \ldots (14).$$

Again, if we have

$$Q = Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy \ldots (15),$$

the equations between which the elimination is to be effected will be

$$\left.\begin{aligned} Ax + Fy + Ez &= 0 \\ Fx + By + Dz &= 0 \\ Ex + Dy + Cz &= 0 \end{aligned}\right\} \ldots (16),$$

whence the result sought becomes, on reduction,

$$\theta(Q) = ABC + 2DEF - (AD^2 + BE^2 + CF^2) \ldots (17).$$

Finally, if Q be of the form,

$$Q = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 \ldots (18),$$

we shall, on taking the first differential coefficients, have

$$Ax^2 + 2Bxy + Cy^2 = 0,$$
$$Bx^2 + 2Cxy + Dy^2 = 0.$$

From these, eliminating x^2 and y^2, and dividing the results by y and by x respectively, we find

$$\left.\begin{array}{l} 2(B^2 - AC)\,x - (AD - BC)\,y = 0 \\ (AD - BC)\,x - 2(C^2 - BD)\,\,y = 0 \end{array}\right\}\ldots.(19),$$

which, in accordance with a previous remark, are linear. Hence, on elimination,

$$\theta(Q) = (AD - BC)^2 - 4\,(B^2 - AC)\,(C^2 - BD)\ldots.(20);$$

and in a similar way may we proceed for more complicated cases.

5. It is evident, to resume our original notation, that the equation, $\theta(Q) = 0$, expresses the relation which must be fulfilled among the constants A_1, A_2, .. A_m, in order that the equations, (7), may admit of being satisfied without the simultaneous vanishing of x_1, x_2, .. x_m. We shall suppose this condition to be fulfilled, and that the equations, (7), are in reality satisfied, while x_1, x_2, .. x_m retain actual values. By the reasoning of sect. 3, it has been shewn that the equations, (8), will also be satisfied, and by inspection of the linear theorems (5), it will be seen, that y_1, y_2 .. y_m cannot all simultaneously vanish, consistently with our assumptions relatively to x_1, x_2, .. x_m; hence it will be necessary that the condition $\theta(R) = 0$, be also satisfied among the constants B_1, B_2, .. B_m. Thus the condition $\theta(Q) = 0$, when satisfied, involves as a necessary consequence, the fulfilment of the analogous condition, $\theta(R) = 0$; and the same relation of mutual dependence exists between $\theta(q)$ and $\theta(r)$.

Now it is scarcely necessary to observe, that the constitution of the two functions, Q and q, relatively to the constants they involve, will not generally be such, as that the conditions, $\theta(Q) = 0$, $\theta(q) = 0$, shall be thereby satisfied. Here therefore we are to inquire, whether it is not possible to obtain from these a third function, which, by the particular determinations of an arbitrary constant, shall enable us to satisfy the conditions required. In order to effect this, add the primitive equations (A_3), (B_3), after having multiplied the former by an indeterminate constant quantity h, we have

$$Q + hq = R + hr\ldots\ldots(21).$$

Like each of the original equations, this will be homogeneous. Considered as the subject of the argument above developed, it leads to the conclusion, that the two systems of equations,

$$\frac{d(Q+hq)}{dx_1}=0, \quad \frac{d(Q+hq)}{dx_2}=0, \ldots \frac{d(Q+hq)}{dx_m}=0 \ldots (22),$$

$$\frac{d(R+hr)}{dy_1}=0, \quad \frac{d(R+hr)}{dy_2}=0, \ldots \frac{d(R+hr)}{dy_m}=0 \ldots (23),$$

are mutually dependent; and as a further consequence of the same process of reasoning, that if the constant h be so determined as to satisfy the equation,

$$\theta(Q+hq)=0 \ldots (24),$$

then will the analogous equation

$$\theta(R+hr)=0 \ldots (25)$$

be satisfied also. I shall now shew that this principle involves the complete solution of the problem under consideration.

6. Let ϕ, as a symbol of functionality, indicate the combinations of the constants in $\theta(Q)$, $\theta(R)$, &c., so that

$$\theta(Q)=\phi(A_1A_2, \ldots A_m), \quad \theta(R)=\phi(B_1B_2, \ldots B_m),$$
$$\theta(q)=\phi(a_1a_2, \ldots a_m), \quad \theta(r)=\phi(b_1b_2, \ldots b_m).$$

Then substituting in (21) the forms assigned in (3), we have

$$(A_1+ha_1)x_1^n+(A_2+ha_2)x_2^n \ldots +(A_m+ha_m)x_m^n+\Sigma(A_\iota+ha_\iota)x_1^\alpha x_2^\beta \ldots x_m^\mu$$
$$=(B_1+hb_1)y_1^n+(B_2+hb_2)y_2^n \ldots +(B_m+hb_m)y_m^n+\Sigma(B_\iota+hb_\iota)y_1^\alpha y_2^\beta \ldots y_m^\mu$$
$$\ldots (26).$$

And the equations (24), (25), on replacing θ by ϕ, thus give

$$\phi\{(A_1+ha_1), \quad (A_2+ha_2), \ldots (A_\nu+ha_\nu)\}=0 \ldots (27),$$
$$\phi\{(B_1+hb_1), \quad (B_2+hb_2), \ldots (B_\nu+hb_\nu)\}=0 \ldots (28).$$

As the values of h satisfying these equations must be identical, and as those values are to be sought by the development of their first members in ascending or descending powers of that quantity, it is evident that those equations, in their developed forms, must be equivalent relatively to that quantity. If we then observe that the terms independent of h in the two developments, are $\phi(A_1, A_2 \ldots A_\nu)$ and $\phi(B_1, B_2 \ldots B_\nu)$ respectively, and that the corresponding coefficients of the highest power of h are $\phi(a_1, a_2 \ldots a_\nu)$ and $\phi(b_1, b_2 \ldots b_\nu)$ respectively, and that the intermediate terms are formed according to the ordinary laws of development by Taylor's theorem, it will be manifest, that in order to establish the proposed equivalence, we must have

$$\frac{\phi(A_1, A_2 \ldots A_\nu)}{\phi(a_1, a_2 \ldots a_\nu)}=\frac{\phi(B_1, B_2 \ldots B_\nu)}{\phi(b_1, b_2 \ldots b_\nu)} \ldots (29),$$

$$\frac{\left(a_1 \frac{d}{dA_1} + a_2 \frac{d}{dA_2} \cdot\cdot + a_\nu \frac{d}{dA_\nu}\right) \phi(A_1, A_2 \ldots A_\nu)}{\phi(a_1\, a_2 \ldots a_\nu)}$$

$$= \frac{\left(b_1 \frac{d}{dB_1} + b_2 \frac{d}{dB_2} \cdot\cdot + b_\nu \frac{d}{dB_\nu}\right) \phi(B_1\, B_2 \ldots B_\nu)}{\phi(b_1\, b_2 \ldots b_\nu)} \ldots (30),$$

$$\frac{\left(a_1 \frac{d}{dA_1} + a_2 \frac{d}{dA_2} \cdot\cdot + a_\nu \frac{d}{dA_\nu}\right)^\gamma \phi(A_1, A_2 \ldots A_\nu)}{\phi(a_1\, a_2 \ldots a_\nu)}$$

$$= \frac{\left(b_1 \frac{d}{dB_1} + b_2 \frac{d}{dB_2} \cdot\cdot + b_\nu \frac{d}{dB_\nu}\right)^\gamma \phi(B_1, B_2 \ldots B_\nu)}{\phi(b_1\, b_2 \ldots b_\nu)} \ldots\ldots (31),$$

if we represent by γ the degree of $\phi(A_1, A_2, \ldots A_\nu)$, which will of course determine the limit of the orders of differentiation. On replacing $\phi(A_1, A_2, \ldots A_\nu)$ by $\theta(Q)$, &c., our equations become

$$\frac{\theta(Q)}{\theta(q)} = \frac{\theta(R)}{\theta(r)} \ldots\ldots (32,)$$

$$\frac{\left(a_1\frac{d}{dA_1}+a_2\frac{d}{dA_2} \cdot\cdot + a_\nu \frac{d}{dA_\nu}\right)\theta(Q)}{\theta(q)} = \frac{\left(b_1\frac{d}{dB_1}+b_2\frac{d}{dB_2} \cdot\cdot + b_\nu\frac{d}{dB_\nu}\right)\theta(R)}{\theta(r)},$$

$$\ldots\ldots (33),$$

$$\frac{\left(a_1\frac{d}{dA_1}+a_2\frac{d}{dA_2} \cdot\cdot + a_\nu \frac{d}{dA_\nu}\right)^\gamma \theta(Q)}{\theta(q)} = \frac{\left(b_1\frac{d}{dB_1}+b_2\frac{d}{dB_2} \cdot\cdot + b_\nu\frac{d}{dB_\nu}\right)^\gamma \theta(R)}{\theta(r)}$$

$$\ldots\ldots (34),$$

which are the simplest forms under which the full solution can be placed, and are a direct consequence of the relations (24) and (25). The results of this part of our inquiry may therefore be comprised in the following general proposition.

A. *If* Q *and* q *represent two similar homogeneous functions of the same degree, which are linearly and similarly transformed into* R *and* r *respectively, and if the symbol* θ *before a proposed homogeneous function be understood to indicate the result of the elimination of the variables, from the first differential coefficients of that function, equated to* 0, *then are the relations among the coefficients of* Q, R, q, r, *the same as are necessary in order to verify the assumed identity of the equations,*

$$\theta(Q + hq) = 0, \quad \theta(R + hr) = 0,$$

relatively to the constant h.

It may be proper to observe, that the analysis on which the above theorem is here made to depend, is considerably different from that by which I originally obtained it. This, in fact, consisted in an extension of the method which I on a former occasion employed, when treating the same subject in the pages of this Journal, vide No. VIII. Vol. II. p. 64.

7. Besides the literal coefficients $A_1, A_2, \ldots A_\nu, B_1, B_2$, &c., it commonly happens, that the functions Q, q, R, r, as in the examples of section 4, are in some of their terms affected with numerical multipliers. Provided however that these multipliers are the same, and are similarly employed, in Q and q, and again in R and r, no change will be thereby introduced in the symbolical forms of the general solution, (32), (33), (34). For let $k_1, k_2 \ldots k_\nu$ be numerical quantities, and let the compound coefficients of the several terms in Q be $k_1 A_1$, $k_2 A_2, \ldots k_\nu A_\nu$, and of those in q taken in the same order $k_1 a_1$, $k_2 a_2, \ldots k_\nu a_\nu$, then will those of $Q+hq$ be

$$k_1(A_1+ha_1),\ k_2(A_2+ha_2) \ldots k_\nu(A_\nu+ha_\nu) \ldots\ldots (35).$$

Now $k_1, k_2, \ldots k_\nu$, being numerical, will not distinctively appear in $\theta(Q)$ and $\theta(q)$ which will therefore, as before, assume the form

$$\phi(A_1 A_2 \ldots A_\nu),\quad \phi(a_1 a_2 \ldots a_\nu).$$

Hence also will $\theta(Q+hq)$ become

$$\theta(Q+hq)=\phi\{(A_1+ha_1)(A_2+ha_2) \ldots (A_\nu+ha_\nu)\} \ldots\ldots (36),$$

and adopting the same class of numerical coefficients with B_1, $B_2, \ldots B_\nu$, in the expression of R and r,

$$\theta(R+hr)=\phi\{(B_1+hb_1)(B_2+hb_2) \ldots (B_\nu+hb_\nu)\} \ldots\ldots (37).$$

These forms are identical with those employed in the previous sections, although the interpretation of ϕ will be modified by the introduction of $k_1, k_2, \ldots k_\nu$. Taken in connexion with the theorem A, they lead to the same series of equations, presented under the general type,

$$\frac{\left(a_1\dfrac{d}{dA_1}+a_2\dfrac{d}{dA_2} \ldots +a_\nu\dfrac{d}{dA_\nu}\right)^\eta \theta(Q)}{\theta(q)}=\frac{\left(b_1\dfrac{d}{dB_1}+b_2\dfrac{d}{dB_2} \ldots +b_\nu\dfrac{d}{dB_\nu}\right)^\eta \theta(R)}{\theta(r)} \ldots\ldots\ldots (38),$$

η being an indefinite integer varying from 0 upwards.

If $\eta=\gamma$ (which is the highest value it can receive without causing both sides to vanish), it is evident, by the nature of developments, that this equation will become

$$\frac{\theta(q)}{\theta(q)}=\frac{\theta(r)}{\theta(r)}, \text{ or } 1=1.$$

Hence the values of η which are to be employed, range from 0 to $\gamma - 1$. Of the series of conditions (32), (33), (34), the uppermost is peculiarly deserving of attention, and it may be shewn that the succeeding ones might be formed, by the extension of A to the whole class of homogeneous equations represented under the type,

$$Q + nq = R + nr,$$

n being entirely arbitrary. It is in fact on inspection evident, that the series of conditions found by the development of both members of the equation

$$\frac{\theta(Q + nq)}{\theta(q)} = \frac{\theta(R + nr)}{\theta(r)}$$

would reproduce the system (32), (33), (34).

8. As a first example of the application of the above theorems, I select the very simple case

$$ax^2 + 2bxy + cy^2 = a'x'^2 + 2b'x'y' + c'y'^2 \ldots\ldots (39),$$

$$Ax^2 + 2Bxy + Cy^2 = A'x'^2 + 2B'x'y' + C'y'^2 \ldots. (40).$$

Here by (14) $\theta(Q) = AC - B^2$, $\theta(q) = ac - b^2$, $\theta(R) = A'C' - B'^2$, $\theta(r) = a'c' - b'^2$, which values are to be employed in the symbolical forms of the general solution (38) as applied to this particular case, viz.

$$\frac{\theta(Q)}{\theta(q)} = \frac{\theta(R)}{\theta(r)}$$

$$\frac{\left(a\dfrac{d}{dA} + b\dfrac{d}{dB} + c\dfrac{d}{dC}\right)\theta(Q)}{\theta(q)} = \frac{\left(a'\dfrac{d}{dA'} + b'\dfrac{d}{dB'} + c'\dfrac{d}{dC'}\right)\theta(R)}{\theta(r)}.$$

This being done, and the requisite differentiations effected, we find

$$\frac{AC - B^2}{ac - b^2} = \frac{A'C' - B'^2}{a'c' - b'^2} \ldots\ldots\ldots (41),$$

$$\frac{aC - 2bB + cA}{ac - b^2} = \frac{a'C' - 2b'B' + c'A'}{a'c' - b'^2} \ldots (42).$$

The equation (39) may be regarded as characteristic of the nature of the transformation to be effected. Should that correspond to the geometrical idea of a change of co-ordinates, from a pair of axes x, y, making an angle θ, to another pair x', y', whose angle of inclination is θ', then will (39) become

$$x^2 + 2xy\cos\theta + y^2 = x'^2 + 2x'y'\cos\theta' + y'^2$$

$$\therefore\ a = 1,\ b = \cos\theta,\ c = 1,\ a' = 1,\ b' = \cos\theta',\ c' = 1.$$

Hence (41) and (42) give

$$\frac{AC - B^2}{(\sin\theta)^2} = \frac{A'C' - B'^2}{(\sin\theta')^2} \text{ . . . . } (43),$$

$$\frac{A - 2B\cos\theta + C}{(\sin\theta)^2} = \frac{A' - 2B'\cos\theta' + C'}{(\sin\theta')^2} \text{ . . } (44).$$

We pass next to the important case

$$\left.\begin{aligned} x^2 + y^2 + z^2 &= x'^2 + y'^2 + z'^2, \\ Ax^2 + By^2 + Cz^2 &+ 2Dyz + 2Exz + 2Fxy \\ = A'x'^2 + B'y'^2 &+ C'z'^2 + 2D'y'z' + 2E'x'z' + 2F'x'y' \end{aligned}\right\} \text{ . . } (45).$$

Here, by reference to (17), we find

$$\theta(Q) = ABC + 2DEF - (AD^2 + BE^2 + CF^2).\theta(q) = 1 \text{ . . . . } (46),$$

$$\theta(R) = A'B'C' + 2D'E'F' - (A'D'^2 + B'E'^2 + C'F'^2).\theta(r) = 1 \text{ . . } (47),$$

which expressions are to be employed in the symbolical forms,

$$\frac{\theta(Q)}{\theta(q)} = \frac{\theta(R)}{\theta(r)},$$

$$\frac{\left(\frac{d}{dA} + \frac{d}{dB} + \frac{d}{dC}\right)\theta(Q)}{\theta(q)} = \frac{\left(\frac{d}{dA'} + \frac{d}{dB'} + \frac{d}{dC'}\right)\theta R}{\theta(r)},$$

$$\frac{\left(\frac{d}{dA} + \frac{d}{dB} + \frac{d}{dC}\right)^2\theta(Q)}{\theta(q)} = \frac{\left(\frac{d}{dA'} + \frac{d}{dB'} + \frac{d}{dC'}\right)^2\theta(R)}{\theta(r)},$$

whence, by mere inspection, we have

$$ABC + 2DEF - (AD^2 + BE^2 + CF^2)$$
$$= A'B'C' + 2D'E'F' - (A'D'^2 + B'E'^2 + C'F'^2) \text{ . . . . } (48),$$

$$AB + BC + AC - (D^2 + E^2 + F^2) = A'B' + B'C' + A'C' - (D'^2 + E'^2 + F'^2) \, (49),$$

$$A + B + C = A' + B' + C' \text{ . . . . } (50).$$

If D' E' F' are supposed to vanish, the above system becomes equivalent to the remarkable cubic, so frequently met with in Analytical Mechanics, and the Geometry of Space.

Were the number of the variables four, the conditions of the problem remaining in other respects unaltered, we should in the same way obtain an equation of the fourth degree, or rather a system of equations thereto equivalent, determining the values of the four constants in R, and so on to any proposed number of variables.

9. Let us now attribute to our primitive equations the more general forms,

$$\left.\begin{aligned} &ax^2 + by^2 + cz^2 + 2dyz + 2exz + 2fxy \\ &= a'x'^2 + b'y'^2 + c'z'^2 + 2d'y'z' + 2e'x'z' + 2f'x'y' \end{aligned}\right\} \text{for } q = r \text{ . . . . } (51),$$

$$\left.\begin{array}{l} Ax^2+By^2+Cz^2+2Dyz+2Exz+2Fxy \\ =A'x'^2+B'y'^2+C'z'^2+2D'y'z'+2E'x'y'+2F'x'y' \end{array}\right\}\text{for } Q=R.\;(52).$$

Here to the values of $\theta(Q)$ and $\theta(R)$ as given in the last example, we must add

$$\theta(q)=abc+2def-(ad^2+be^2+cf^2).\ \theta(r)=a'b'c'+2d'e'f'-(a'd'^2+b'e'^2+c'f'^2).$$

The symbolical forms of the solution for this case, are at once seen to be

$$\frac{\theta(Q)}{\theta(q)}=\frac{\theta(R)}{\theta(r)}\;\ldots\ldots\ldots\ldots\ldots\ldots\;(53),$$

$$\frac{\left(a\frac{d}{dA}+b\frac{d}{dB}\ldots+f\frac{d}{dF}\right)\theta(Q)}{\theta(q)}=\frac{\left(a'\frac{d}{dA'}+b'\frac{d}{dB'}\ldots+f'\frac{d}{dF'}\right)\theta(R)\ldots}{\theta(r)}\;(54)$$

$$\frac{\left(a\frac{d}{dA}+b\frac{d}{dB}\ldots+f\frac{d}{dF}\right)^2\theta(Q)}{\theta(q)}=\frac{\left(a'\frac{d}{dA'}+b'\frac{d}{dB'}+f'\frac{d}{dF'}\right)^2\theta(R)\ldots}{\theta(r)}\;(55).$$

When the algebraic development of (54) is determined, that of (55) will be found, by simply changing in the former a, b, c, &c., into A, B, C, &c., and *vice-versa*. In exhibiting the results of these developments, it will be convenient to represent the differential coefficients of $\theta(Q)$, $\theta(R)$ &c., by subsidiary quantities. Assume therefore

$$L=BC-D^2,\qquad M=AC-E^2,\qquad N=AB-F^2),$$
$$S=2(EF-AD),\quad T=2(DF-BC),\quad U=2(DE-CF),$$
$$L'=B'C'-D'^2,\ \&c.,\quad l=bc-d^2,\ \&c.,\qquad l'=b'c'-d'^2,\ \&c.$$

Then will (53), (54), (55), give, on effecting the operations indicated,

$$\frac{ABC+2DEF-(AD^2+BE^2+CF^2)}{abc+2def-(ad^2+be^2+cf^2)}$$
$$=\frac{A'B'C'+2D'E'F'-(A'D'^2+B'E'^2+C'F'^2)}{a'b'c'+2d'e'f'-(a'd'^2+b'e'^2+c'f'^2)}\;\ldots(56),$$

$$\frac{aL+bM+cN+dS+eT+fU}{abc+2def-(ad^2+be^2+cf^2)}$$
$$=\frac{a'L'+b'M'+c'N'+d'S'+e'T'+f'U'}{a'b'c'+2d'e'f'-(a'd'^2+b'e'^2+c'f'^2)}\;\ldots(57),$$

$$\frac{Al+Bm+Cn+Ds+Et+Fu}{abc+2def-(ad^2+be^2+cf^2)}$$
$$=\frac{A'l'+B'm'+C'n'+D's'+E't'+F'u'}{a'b'c'+2d'e'f'-(a'd'^2+b'e'^2+c'f'^2)}\;\ldots(58).$$

If we wish by the above transformation to represent a change of co-ordinates, from axes x, y, z, given in relative position by the equations $\cos yz = \cos\phi$, $\cos xz = \cos\psi$, $\cos xy = \cos\chi$, to another system of axes x', y', z', whose inclinations are similarly determined by the angles ϕ', ψ', χ', then will our first equation (51) become

$$\left.\begin{aligned} &x^2 + y^2 + z^2 + 2yz\cos\phi + 2xz\cos\psi + 2xy\cos\chi \\ &= x'^2 + y'^2 + z'^2 + 2y'z'\cos\phi' + 2x'z'\cos\psi' + 2x'y'\cos\chi' \end{aligned}\right\} \ldots (59),$$

so that it will only be necessary in the formulæ of solution (56), (57), (58), to make

$$a = b = c = 1,\quad d = \cos\phi,\quad e = \cos\psi,\quad f = \cos\chi,$$
$$a' = b' = c' = 1,\quad d' = \cos\phi',\quad e' = \cos\psi',\quad f' = \cos\chi',$$
$$l = (\sin\phi)^2,\quad m = (\sin\psi)^2,\quad n = (\sin\chi)^2,\ \&c.$$

in order to obtain the relations sought.

10. In further illustration of the general method, let us now take an example of the transformation of homogeneous functions of the third degree, our primitive equations being placed under the forms,

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = a'x'^3 + 3b'x'^2y' + 3c'x'y'^2 + d'y'^3 \ldots (60),$$
$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 = A'x'^3 + 3B'x'^2y' + 3C'x'y'^2 + D'y'^3 \ldots (61).$$

Here by (19) we have

$$\theta(Q) = (AD - BC)^2 - 4(B^2 - AC)(C^2 - BD).$$
$$\theta(q) = (ad - bc)^2 - 4(b^2 - ac)(c^2 - bd),$$
$$\theta(R) = (A'D' - B'C')^2 - 4(B'^2 - A'C')(C'^2 - B'D').$$
$$\theta(r) = (a'd' - b'c')^2 - 4(b'^2 - a'c')(c'^2 - b'd'),$$

which, as in former cases, are to be substituted in the general symbolical forms,

$$\frac{\theta(Q)}{\theta(q)} = \frac{\theta(R)}{\theta(r)} \ldots (62),$$

$$\frac{\left(a\dfrac{d}{dA} + b\dfrac{d}{dB} + c\dfrac{d}{dC} + d\dfrac{d}{dD}\right)\theta(Q)}{\theta(q)} = \frac{\left(a'\dfrac{d}{dA'} + b'\dfrac{d}{dB'} + c'\dfrac{d}{dC'} + d'\dfrac{d}{dD'}\right)\theta(R)}{\theta(r)} \ldots (63),$$

$$\frac{\left(a\dfrac{d}{dA} + b\dfrac{d}{dB} + c\dfrac{d}{dC} + d\dfrac{d}{dD}\right)^3\theta(Q)}{\theta(q)} = \frac{\left(a'\dfrac{d}{dA'} + b'\dfrac{d}{dB'} + c'\dfrac{d}{dC'} + d'\dfrac{d}{dD'}\right)^3\theta(R)}{\theta(r)} \ldots (64).$$

The requisite differentiations being performed, we shall, as in the examples already given, be in possession of the final algebraic relations among the constants of (60) and (61);

relations which, in this case, will evidently be of a somewhat complicated character.

Instead however of employing the above method, we may, by the direct application of our fundamental proposition, A, demonstrated in §. 6, obtain at once a result conveniently adapted for numerical computation. For this purpose, having multiplied the upper equation (60) by h, and to the result added the lower (61), let

$$A + ha = A_1,\ B + hb = B_1, \&c.,\ A' + ha' = A'_1, \&c. \ldots. (65),$$

so that the resulting equation, $Q + hq = R + hr$, may assume the form

$$A_1x^3+3B_1x^2y+3C_1xy^2+D_1y^3=A_1'x'^3+3B_1'x'^2y'+3C_1'x'y'^2+D_1'y'^3 \ldots (66).$$

Then, by the proposition in question, the two final equations

$$(A_1D_1 - B_1C_1)^2 - 4(B_1^2 - A_1C_1)(C_1^2 - B_1D_1) = 0 \ldots (67),$$

$$(A_1'D_1' - B_1'C_1')^2 - 4(B_1'^2 - A_1'C_1')(C_1'^2 - B_1'D_1') = 0 \ldots (68),$$

must be identical relatively to h.

Suppose, for example, it were required to determine whether the equations

$$x^3 - y^3 = 3x'^2y' + 3x'y'^2 + y'^3 \ldots. (69),$$

$$x^3 - xy^2 = 2x'^2y' + 3x'y'^2 + y'^3 \ldots. (70),$$

are derivable from a common system of linear relations, connecting x and y with x' and y'. Here $(69) \times h + (70)$ gives

$$(1 + h)x^3 - xy^2 - hy^3 = (2 + 3h)x'^2y' + 3(1 + h)x'y'^2 + (1 + h)y'^3,$$

whence, by comparison with (66),

$$A_1 = 1 + h.\ B_1 = 0.\ C_1 = -\tfrac{1}{3}.\ D_1 = -h\ A_1' = 0.\ B_1' = \frac{2+3h}{3}\ C_1' = D_1' = 1 + h.$$

The substitution of the above values of A_1, B_1, C_1, D_1, in (67), gives

$$h^4 + 2h^3 + h^2 - \frac{4}{27}h - \frac{4}{27} = 0 \ldots. (71).$$

Again, substituting for A_1', B_1', C_1', D_1', in (68), we obtain on reduction the very same equation, viz.

$$h^4 + 2h^3 + h^2 - \frac{4}{27}h - \frac{4}{27} = 0 \ldots. (72),$$

and from the identity of these results infer, that our primitive equations, (69) and (70), are in reality derived from a common system of linear transformations.

11. The determination of the actual values of the constants involved in the linear theorems, connecting the two sets of variables, constitutes a separate branch of our general investiga-

tions. In proceeding to the discussion of this part of the subject, it will be necessary to resume the notation adopted in the first sections of this memoir.

On referring to §. 4, the reader will perceive, that in the last stage but one of the process of elimination, by which we arrive at $\theta(Q)$, we pass through two equations involving two variables. Those equations, it is there observed, and the truth of the remark might easily be proved, are linear. Applying this observation to the process by which $\theta(Q + hq)$ might be similarly obtained, we see, that in whatever order the elimination is effected, we, in the last stage but one, meet with two linear equations, involving the two variables which are last eliminated. Of these equations, it is however clear that one only can be independent, in consequence of the relation $\theta(Q + hq) = 0$, which we here suppose to be satisfied. Now as the order in which the variables are eliminated is indifferent, so that any two of them may be left as the subjects for the linear relations above mentioned, it is evident that in the whole there must exist $m - 1$ such relations, connecting linearly, and by independent pairs, the m variables, $x_1, x_2, . . x_m$. These relations may be put under the form

$$\frac{x_1}{l_1} = \frac{x_2}{l_2} . . = \frac{x_m}{l_m} . . . . (73),$$

$l_1, l_2, . . l_m$, being functions of the coefficients of the several terms in $(Q + hq)$. Thus may the m independent equations

$$\frac{d(Q + hq)}{dx_1} = 0, \ \frac{d(Q + hq)}{dx_2} = 0 . ., \frac{d(Q + hq)}{dx_m} = 0 . . . . (74),$$

be considered as having merged into the m independent equation

$$\frac{x_1}{l_1} = \frac{x_2}{l_2} . . = \frac{x_m}{l_m}, \ \theta(Q + hq) = 0 \ . . . . (75).$$

Similarly, in the process of elimination, may the m independent equations

$$\frac{d(R + hr)}{dy_1} = 0, \ \frac{d(R + hr)}{dy_2} = 0 . ., \ \frac{d(R + hr)}{dy_m} = 0 . . . . (76),$$

be regarded as merging into an equal number of independent equations,

$$\frac{y_1}{n_1} = \frac{y_2}{n_2} . . = \frac{y_m}{n_m}, \ \theta(R + hr) = 0 \ . . . . (77),$$

$n_1 \, n_2 . . n_m$ being functions of the coefficients in $R + hr$. Now the two systems of equations (74) and (76) are mutually dependent, hence are also the systems (75) and (77). The conse-

quences which follow from the mutual dependence of the two last equations of these two systems have already been examined. The remaining ones, in their mutual dependence, are scarcely of less importance, as enabling us to determine the linear theorems connecting $x_1, x_2 \ldots x_m$ with $y_1, y_2 \ldots y_m$.

12. We see, in fact, that when the values of $x_1, x_2 \ldots x_m$, are chosen proportionals to $l_1, l_2 \ldots l_m$, respectively, the simultaneous values of $y_1, y_2 \ldots y_m$, will be proportional to $n_1, n_2 \ldots n_m$. To determine the actual magnitudes of the values of $y_1, y_2 \ldots y_m$, corresponding to an assumed series of values of $x_1, x_2 \ldots x_m$, another equation is evidently necessary, and for this purpose either of the primitive equations, $q = r$, or $Q = R$, is sufficient. We choose the former, and suppose that the substitution in q of $l_1, l_2 \ldots l_m$, in the place of $x_1, x_2 \ldots x_m$, gives a result L; and that the substitution of $n_1, n_2 \ldots n_m$, for $y_1, y_2 \ldots y_m$, in r, gives N; then it is manifest, that since q and r are homogeneous and of the n^{th} degree, the equation $q = r$ will be satisfied, as well as the $m-1$ first equations of (75) and of (77), by the assumptions

$$x_1 = \frac{l_1}{\sqrt[n]{L}},\quad x_2 = \frac{l_2}{\sqrt[n]{L}} \ldots\ldots x_m = \frac{l_m}{\sqrt[n]{L}} \ldots\ldots (78),$$

$$y_1 = \frac{n_1}{\sqrt[n]{N}},\quad y_2 = \frac{n_2}{\sqrt[n]{N}} \ldots\ldots y_m = \frac{n_m}{\sqrt[n]{N}} \ldots\ldots (79),$$

which are therefore a set of simultaneous values of the $2m$ variables in question.

Now $l_1, l_2 \ldots l_m, n_1, n_2 \ldots n_m$, involve h; as many different values as are therefore assigned to that quantity, so many sets of simultaneous values will the above equations (78) and (79) afford for the variables $x_1, x_2 \ldots x_m, y_1, y_2, \ldots y_m$; the successive substitution of which in the linear forms

$$\left.\begin{aligned} x_1 &= \lambda_1 y_1 + \lambda_2 y_2 \ldots + \lambda_m y_m \\ x_2 &= \mu_1 y_1 + \mu_2 y_2 \ldots + \mu_m y_m \\ &\cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\ x_m &= \rho_1 y_1 + \rho_2 y_2 \ldots + \rho_m y_m \end{aligned}\right\} \ldots\ldots (80),$$

will give equations serving to determine by linear elimination, the values of the constants

$$\lambda_1, \mu_1, \rho_1 \ldots \lambda_m, \mu_m \ldots \rho_m.$$

The application of this method to (69) and (70), leads to the results

$$x = x' + y',$$

$$y = x',$$

which are easily verified. The values of h, for this case, as determined by the solution of (71), are -1, $+\frac{1}{3}$, $-\frac{2}{3}$; and the formulæ to be employed in conjunction with either of the primitive equations (69) and (70), are

$$\frac{x}{A_1D_1 - B_1C_1} = \frac{y}{2(B_1^2 - A_1C_1)} \ldots\ldots (81),$$

$$\frac{x'}{A_1'D_1' - B_1'C_1'} = \frac{y'}{2(B_1'^2 - A_1'C_1')} \ldots\ldots (82),$$

as will be seen on reference to (19).

It might be presumed, that as the values of h are in some cases more numerous, and in others, from equality of the roots, less so, than would appear to be necessary for the formation of the different sets of simultaneous values of the variables to be employed in the above process, we should in the former case arrive at superfluous results, and in the latter be compelled to have recourse to a different method of solution, for the discovery of the linear relations. How far these anticipations might prove correct I am not prepared to say, but I apprehend that under either of these circumstances, as well as under the supposed condition of h receiving imaginary values, an answering peculiarity will be found in the relations sought, rendering the solution possible and definite.

13. The functions $\theta(Q)$ and $\theta(R)$ may be shewn to possess many remarkable properties, both individually and in mutual relation. Of these the one I am about to demonstrate is perhaps the most important. It has been established in this paper, that when any two homogeneous functions, Q and q, with the same variables, and of the same degree, are by a common system of linear relations transformed into R and r, then

$$\frac{\theta(Q)}{\theta(q)} = \frac{\theta(R)}{\theta(r)} \ldots\ldots\ldots (83).$$

Let the ratio of $\theta(R)$ to $\theta(Q)$ be represented by E, so that $\theta(Q) = \frac{\theta(R)}{E}$; then by (83) also $\theta(q) = \frac{\theta(r)}{E}$. The nature of the function E it will be necessary to examine.

It is in the first place evident that E cannot in any way functionally depend on the constants in Q and R, or in q and r, otherwise the equations

$$\theta(Q) = \frac{\theta(R)}{E}, \quad \theta(q) = \frac{\theta(r)}{E},$$

would suppose a relation among the constants in Q and q, which are entirely independent. Hence E can only involve the constants contained in the linear theorems. We proceed to determine its form in one or two simple cases.

Let Q_2 be a homogeneous function of the second degree, with two variables, x and y; and let the linear relations by which Q_2 is supposed to be transformed into R_2, be

$$\left.\begin{aligned} x &= mx' + ny' \\ y &= m'x' + n'y' \end{aligned}\right\} \dots\dots\dots\dots (84).$$

Since E is independent of the constants in Q_2, we may assume the values of those constants at pleasure, provided that our assumptions do not cause $\theta(Q_2)$ to vanish. Let then $Q_2 = x^2 + y^2$, whence $\theta(Q_2) = 1$; also by substitution of (84),

$$R_2 = (m^2 + m'^2)\,x'^2 + 2\,(mn + m'n')\,x'y' + (n^2 + n'^2)\,y'^2,$$

$$\therefore\ \theta(R_2) = (m^2 + m'^2)(n^2 + n'^2) - (mn + m'n')^2 = (mn' - m'n)^2,$$

$$E = \frac{\theta(R_2)}{\theta(Q_2)} = (mn' - m'n)^2.$$

Hence if Q_2 be any homogeneous function of the second degree, transformed by (84) to R_2, we shall always have

$$\theta(Q_2) = \frac{\theta(R_2)}{(mn' - m'n)^2} \dots\dots\dots (85).$$

If Q_3 be a homogeneous function of the third degree, similarly transformed by (84) into R_3, we shall, after an analogous but very complicated process, find

$$\theta(Q_3) = \frac{\theta(R_3)}{(mn' - m'n)^6} \dots\dots\dots (86).$$

The singularity of these results has led me to investigate the general law on which they depend, and I have arrived at the following theorem.

B. *If* Q_n *be a homogeneous function of the* n^{th} *degree, with* m *variables, which by the linear theorems* (80) *is transformed into* R_n, *a similar homogeneous function; and if* γ *represent the degree of* $\theta(Q_n)$ *and* $\theta(R_n)$, *then*

$$\theta(Q_n) = \frac{\theta(R_n)}{E^{\frac{\gamma n}{m}}} \dots\dots\dots\dots (87),$$

E being the result obtained by the elimination of the variables from the second members of the linear theorems (80), equated to 0.

From (85) and (86) we derive the important theorem,

$$\frac{\theta(Q_3)}{\{\theta(Q_2)\}^3} = \frac{\theta(R_3)}{\{\theta(R_2)\}^3} \dots\dots (88),$$

a theorem which, by (87), is easily extended to general indices. The application of this result we shall have occasion to exemplify in the researches with which the second part of this memoir will be occupied.

Resuming our former notation, we may observe from (87), that if $E = 0$, then either $\theta(Q)$ and $\theta(q)$ become infinite, or $\theta(R)$ and $\theta(r)$ vanish. Under either of these circumstances, the equations of the general solution (32), (33), (34), disappear, or are greatly modified. By the nature of linear elimination it is evident, that the condition $E = 0$ and the condition $F(\lambda_1, \mu_1, \rho_1 \dots \lambda_m, \mu_m, \rho_m) = 0$ of (9), are identical, a circumstance which verifies the remark in 83, relative to the latter condition.

Minster Yard, Lincoln, April 28*th*, 1841.

II.—ON THE SURD ROOTS OF EQUATIONS.

By R. Moon, M.A. Fellow of Queens' College.

Lemma. If we have the equation

$$A + Bx^{\frac{1}{m}} + Cx^{\frac{2}{m}} + \dots + Lx^{\frac{m-1}{m}} = 0,$$

where $A, B, C \dots L$ and x are all rational quantities, and $x^{\frac{1}{m}}, x^{\frac{2}{m}} \dots x^{\frac{m-1}{m}}$, are all surds, we must have

$$A = 0, \quad B = 0, \quad C = 0, \dots L = 0.$$

Suppose the rule to apply to the equation

$$a + a_1x^{\frac{1}{m}} + a_2x^{\frac{2}{m}} + \dots + a_{m-2}x^{\frac{m-2}{m}} = 0 \quad \dots\dots (1),$$

it must then also apply to the equation

$$a + a_1x^{\frac{1}{m}} + a_2x^{\frac{2}{m}} + \dots + a_{m-2}x^{\frac{m-2}{m}} + a_{m-1}x^{\frac{m-1}{m}} = 0.$$

For from this last we have

$$aa_{m-2} + a_1a_{m-2}x^{\frac{1}{m}} + a_2a_{m-2}x^{\frac{2}{m}} + \dots + a^2_{m-2}x^{\frac{m-2}{m}} + a_{m-2}a_{m-1}x^{\frac{m-1}{m}} = 0,$$

and

$$aa_{m-1}x^{\frac{1}{m}} + a_1a_{m-1}x^{\frac{2}{m}} + a_2a_{m-1}x^{\frac{3}{m}} + \dots + a_{m-2}a_{m-1}x^{\frac{m-1}{m}} + a^2_{m-1}x = 0,$$

or subtracting,

$$\left.\begin{array}{l} aa_{m-2} - a^2_{m-1}x + (a_1a_{m-2} - aa_{m-1})\,x^{\frac{1}{m}} + (a_2a_{m-2} - a_1a_{m-1})\,x^{\frac{2}{m}} \\ \qquad + \&\text{c.} \\ \qquad + (a^2_{m-2} - a_{m-3}a_{m-1})\,x^{\frac{m-2}{m}} \end{array}\right\} = 0 \quad . . (2);$$

whence by our hypothesis we have

$$\begin{aligned} aa_{m-2} - a^2_{m-1}x &= 0, \\ a_1a_{m-2} - aa_{m-1} &= 0, \\ a_2a_{m-2} - a_1a_{m-1} &= 0, \\ \dots\dots\dots\dots &= 0, \\ a^2_{m-2} - a_{m-3}a_{m-1} &= 0; \end{aligned}$$

from which series of equations we obtain, by eliminating $a, a_1, a_2 \dots a_{m-3}$,

$$x = \left(\frac{a_{m-2}}{a_{m-1}}\right)^m,$$

an equation which comprises all the roots of equation (2).

Now it is evident that if we eliminate $x^{\frac{m-1}{m}}$ from the equation

$$a - e\left(\frac{a_{m-2}}{a_{m-1}}\right)^m + ex + a_1x^{\frac{1}{m}} + a_2x^{\frac{2}{m}} + \dots + a_{m-1}x^{\frac{m-1}{m}} = 0,$$

(which we will represent by $u = 0$,) by the same process which we adopted with respect to (1), that we shall in like manner arrive at the result

$$x = \left(\frac{a_{m-2}}{a_{m-1}}\right)^m \dots\dots\dots (3);$$

and this last equation will include all the roots of

$$u(a_{m-2} - a_{m-1}x^{\frac{1}{m}}) = 0,$$

and therefore of $u = 0$.

If we represent the roots of the equation $x^m - 1 = 0$, by $1, \alpha_1, \alpha_2, \dots \alpha_{m-1}$, it is clear that u will be divisible by

$$A = \left(x^{\frac{1}{m}} - \alpha_1\frac{a_{m-2}}{a_{m-1}}\right)\left(x^{\frac{1}{m}} - \alpha_2\frac{a_{m-2}}{a_{m-1}}\right) \dots \left(x^{\frac{1}{m}} - \alpha_{m-1}\frac{a_{m-2}}{a_{m-1}}\right),$$

without a remainder. Now

$$u = e \,.\left(x - \frac{a_{m-2}}{a_{m-1}}\right)^m + \phi\left(x^{\frac{1}{m}}\right),$$

where ϕ represents a function of $m-1$ dimensions; and if we divide $e\,.\left(x - \frac{a_{m-2}}{a_{m-1}}\right)^m$ by A, the result is $\left(x^{\frac{1}{m}} - a\right) e$; whence is plain that $\phi\left(x^{\frac{1}{m}}\right) = A \times \text{const.} = Aa_{m-1}$; hence the roots of the equation $u = 0$ are

$$\left(e \,.\, \frac{a_{m-2}}{a_{m-1}} - a_{m-1}\right)\left(a_1 \frac{a_{m-2}}{a_{m-1}}\right)\left(a_2 \frac{a_{m-2}}{a_{m-1}}\right) \dots \left(a_{m-1} \frac{a_{m-2}}{a_{m-1}}\right).$$

But we have before seen that the roots of $u = 0$ are included among the roots of (3), therefore $\left(e \,.\, \frac{a_{m-2}}{a_{m-1}} - a_{m-1}\right)$ must be identical with some one of the quantities

$$\frac{a_{m-2}}{a_{m-1}}, \quad a_1 \frac{a_{m-2}}{a_{m-1}}, \quad a_2 \frac{a_{m-2}}{a_{m-1}}, \dots a_{m-1} \frac{a_{m-2}}{a_{m-1}},$$

which is impossible, since e may be any rational quantity whatever.

Suppose the rule to apply to the equation

$$a + a_1 x^{\frac{1}{m}} + a_2 x^{\frac{2}{m}} + \dots + a_{m-3} x^{\frac{m-3}{m}} = 0,$$

it will also apply to the equation

$$a + a_1 x^{\frac{1}{m}} + a_2 x^{\frac{2}{m}} + \dots + a_{m-3} x^{\frac{m-3}{m}} + a_{m-2} x^{\frac{m-2}{m}} = 0 \dots\dots (4).$$

For from this last we have

$$aa_{m-3} x^{\frac{1}{m}} + a_1 a_{m-3} x^{\frac{2}{m}} + \dots + a^2_{m-3} x^{\frac{m-2}{m}} + a_{m-2} a_{m-3} x^{\frac{m-1}{m}} = 0,$$

and $\quad aa_{m-2} x^{\frac{2}{m}} + a_1 a_{m-2} x^{\frac{3}{m}} + \dots + a_{m-2} a_{m-3} x^{\frac{m-1}{m}} + a^2_{m-2} x \quad = 0,$

and subtracting we have

$$\left.\begin{array}{l} - a^2_{m-2} x + aa_{m-3} x^{\frac{1}{m}} + (a_1 a_{m-3} - aa_{m-2})\, x^{\frac{2}{m}} \\ \qquad + \dots \\ \qquad + (a^2_{m-3} - a_{m-2} a_{m-4})\, x^{\frac{m-2}{m}} \end{array}\right\} = 0;$$

and eliminating $x^{\frac{m-2}{m}}$ from this last by means of (4), (which we will represent by $u = 0$,) the resulting equation will be

$$u\,(a_{m-3}-a_{m-2}x^{\frac{1}{3}})\,x^{\frac{1}{3}}\times\text{const.}=0\ldots(5);$$

whence, by means of our hypothesis, it is plain we can find

$$x=\phi\,(a_{m-2}a_{m-3})\ldots\ldots\ldots\ldots\ldots(6),$$

an equation whose roots comprise all the finite roots of (5), and which gives m different values of $x^{\frac{1}{m}}$; whence it follows that the equation $u=0$ must have $m-1$ different roots, which is absurd.

It may be proved in a manner precisely similar to that adopted in the latter of the two above cases, that if the rule applies to the equation

$$a+a_1x^{\frac{1}{m}}+a_2x^{\frac{2}{m}}+\ldots\ldots+a_{m-n}{}^{\frac{m-n}{m}}=0,$$

it will also apply to the equation

$$a+a_1x^{\frac{1}{m}}+a_2x^{\frac{2}{m}}+\ldots\ldots+a_{m-n}x^{\frac{m-n}{m}}+a_{m-n+1}x^{\frac{m-n_1+1}{m}}=0.$$

But the rule holds for the equation

$$a+a_1x^{\frac{1}{m}}=0;$$

hence it holds generally.

We are now enabled to shew that, if $a+b^{\frac{1}{m}}$ be a root of an equation, where a and b are rational, but $b^{\frac{1}{m}}$ is a surd; $a+\alpha b^{\frac{1}{m}}$ is also a root where α is any root of the equation $x^m-1=0$.

Take the case of

$$x^5+px^4+qx^3+rx^2+sx+t=0.$$

and let the result of the substitution of $a+b^{\frac{1}{3}}$ in this equation be

$$b^{\frac{5}{3}}+A_1b^{\frac{4}{3}}+A_2b+A_3b^{\frac{2}{3}}+A_4b^{\frac{1}{3}}+A_5=0;$$

whence, by our Lemma,

$$b^{\frac{5}{3}}+A_3b^{\frac{2}{3}}=0\ldots\ldots\ldots(1),$$

$$A_1b^{\frac{4}{3}}+A_4b^{\frac{1}{3}}=0\ldots\ldots\ldots(2),$$

$$A_2b+A_5=0\ldots\ldots\ldots(3).$$

Let α be a root of the equation $x^3-1=0$; then we have

$$(1)\times\alpha^2+(2)\times\alpha+(3)=0,$$

or (observing that $\alpha^3 = 1$,) we have, making the necessary arrangement,

$$b^{\frac{5}{3}}\alpha^5 + A_1 b^{\frac{4}{3}}\alpha^4 + A_2 b\alpha^3 + A_3 b^{\frac{2}{3}}\alpha^2 + A_4 b^{\frac{1}{3}}\alpha + A_5 = 0;$$

that is, $a + \alpha b^{\frac{1}{m}}$ is a root; and the same proof evidently applies generally. Hence, if an equation has a root $a + b^{\frac{1}{m}}$, it has a corresponding factor

$$(x - a)^m - b = 0.$$

It may be observed, that the occurrence of these groups of roots will, in certain cases, facilitate the solution: thus in the present case the solution might be effected by means of an equation of four dimensions.

It will be found also, that if m be the denominator of the surd, and n the number of dimensions of the equation, and

$$m > \frac{n+1}{2}, \text{ where } n \text{ is odd,}$$

$$\text{or } m > \frac{n}{2}, \text{ where } n \text{ is even;}$$

the group of roots $a + b^{\frac{1}{m}}$, will introduce no more complexity than a single root; that is, the equation may be solved by means of one of $n - m + 1$ dimensions.

It only remains to add, that if

$$a + b^{\frac{1}{2}} + c^{\frac{1}{3}}$$

be a root of an equation, there will be six roots depending on the same irrational parts, which are comprised under the form

$$a \pm b^{\frac{1}{2}} + \alpha c^{\frac{1}{3}},$$

where α is a root of the equation $x^3 - 1 = 0$; for whatever be the number of roots depending upon $b^{\frac{1}{2}}$, $c^{\frac{1}{3}}$, it is clear that if when $b = 0$, a root $a + c^{\frac{1}{3}}$ occur, we must have corresponding to it two others, $a + \alpha c^{\frac{1}{3}}$, $a + \alpha^2 c^{\frac{1}{3}}$; hence, if a root occur in the proposed equation, of the form $a + c^{\frac{1}{3}} + \phi$, where ϕ vanishes when $b = 0$, there must be corresponding to it the roots $a + \alpha c^{\frac{1}{3}} + \phi$, $a + \alpha^2 c^{\frac{1}{3}} + \phi$. In like manner it may be shewn that whenever a root $a + b^{\frac{1}{2}} + \phi$ occurs in the pro-

posed equation where ϕ vanishes when $c^{\frac{1}{3}} = 0$ there must be corresponding to it the root $a - b^{\frac{1}{2}} + \phi$. Hence, corresponding to

$$a + b^{\frac{1}{2}} + c^{\frac{1}{3}}, \text{ we have } a + b^{\frac{1}{2}} + ac^{\frac{1}{3}},$$

$$a + b^{\frac{1}{2}} + a^2c^{\frac{1}{3}};$$

corresponding to

$$a + b^{\frac{1}{2}} + ac^{\frac{1}{3}}, \text{ we have } a - b^{\frac{1}{2}} + ac^{\frac{1}{3}},$$

$$a + b^{\frac{1}{2}} + a^2c^{\frac{1}{3}}, \; \ldots\ldots\ldots \; a - b^{\frac{1}{2}} + a^2c^{\frac{1}{3}}.$$

And generally, if we have an equation having a root

$$a + b^{\frac{1}{m}} + c^{\frac{1}{n}} + d^{\frac{1}{p}} + \ldots.$$

it must have for a root

$$a + \beta b^{\frac{1}{m}} + \gamma c^{\frac{1}{n}} + \delta d^{\frac{1}{p}} + \ldots.$$

where β is any root of the equation $x^m - 1 = 0$,

γ .... $x^n - 1 = 0$,

δ .... $x^p - 1 = 0$,

.... $= 0$,

the number of roots in the last case will be $= m.n.p\ldots.$ and it may easily be shewn that the factor to which they will jointly give rise will be rational.

III.—NOTE ON A PASSAGE IN FOURIER'S HEAT.*

In finding the motion of heat in a sphere, Fourier expands a function Fx, arbitrary between the limits $x = 0$ and $x = X$, in a series of the form

$$a_1 \sin n_1x + a_2 \sin n_2x + \&c.$$

where n_1, n_2, &c. are the successive roots of the equation

$$\frac{\tan nX}{nX} = 1 - hX.$$

Now Fourier gives no demonstration of the possibility of this

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* From a Correspondent.

expansion, but he merely determines what the coefficients a_1, a_2, &c. would be, if the function were represented by a series of this form. Poisson arrives, by another method, at the same conclusion as Fourier, and then states this objection to Fourier's solution; but, as is remarked by Mr. Kelland, (*Theory of Heat*, p. 81, Note,) he "does not appear, as far as I can see, to get over the difficulty." The writer of the following article hopes that the demonstration in it will be considered as satisfactory, and consequently as removing the difficulty.

$$\text{Let} \quad n_i X = \epsilon_i, \quad \frac{\pi x}{X} = x', \quad \text{and } Fx = fx'.$$

Then the preceding series will take the form

$$a_1 \sin \frac{\epsilon_1 x}{\pi} + a_2 \sin \frac{\epsilon_2 x}{\pi} + \&c.,$$

the accents being omitted above x.

Now it is shewn by Fourier, that

$$\epsilon_i = \left(\frac{2i-1}{2} - c_i\right)\pi,$$

where c_i is always less than $\frac{1}{2}$, and is equal to 0, when i is infinitely great. Hence the series becomes

$$a_1 \sin\left(\frac{1}{2} - c_1\right)x + a_2 \sin\left(\frac{3}{2} - c_2\right)x + \&c. \ldots . (a).$$

Now it is easily shown, from the fact that any function of x can be represented, between the limits 0 and π, by a series of either sines or cosines of multiples of x, that it may be represented, between the same limits, by a series of the form

$$A \sin \frac{1}{2} x + B \sin \frac{3}{2} x + \&c.$$

Hence each of the quantities

$$\sin\left(\frac{1}{2} - c_1\right)x, \quad \sin\left(\frac{3}{2} - c_2\right)x, \ \&c.,$$

can be developed in a series of this form. We may consequently assume $\sin(\frac{1}{2} - c_1)x$ equal i terms of a series of sines of odd multiples of $\frac{1}{2}x$, together with a quantity, ${}^ie_1$; $\sin\left(\frac{3}{2} - c_2\right)x$ equal to i terms of a similar series, together with a quantity ${}^ie_2$; and so with all the terms of the series (a), up to the term $\sin\left(\frac{2i-1}{2} - c_i\right)x$, which may be assumed

equal to i terms, together with a quantity ${}^ie_i$; and it is readily seen, that each of the quantities ${}^ie_1$, ${}^ie_2$, ... ${}^ie_i$, is infinitely small when i is infinitely great. Hence we shall have

$$a_1 \sin\left(\frac{1}{2}-c_1\right)x + a_2 \sin\left(\frac{3}{2}-c_2\right)x + \ldots + a_i \sin\left(\frac{2i-1}{2}-c_i\right)x$$

$$= A_1 \sin\frac{1}{2}x + A_2 \sin\frac{3}{2}x + \ldots + A_i \sin\frac{2i-1}{2}x$$

$$+ a_1{}^ie_1 + a_2{}^ie_2 + \ldots + a_i{}^ie_i,$$

A_1, A_2, ... A_i, being known, in terms of a_1, a_2, ... a_i. Hence, conversely, any series,

$$A_1 \sin\frac{1}{2}x + A_2 \sin\frac{3}{2}x + \ldots + A_i \sin\frac{2i-1}{2}x,$$

where A_1, A_2, ... A_i, are arbitrary, may be represented by another series of the form

$$a_1\left\{\sin\left(\frac{1}{2}-c_1\right)x - {}^ie_1\right\} + a_2\left\{\sin\left(\frac{3}{2}-c_2\right)x - {}^ie_2\right\} + \ldots$$

$$+ a_i\left\{\sin\left(\frac{2i-1}{2}x - c_i\right) - {}^ie_i\right\},$$

where $a_1, a_2, \ldots a_i$ are determined, in terms of $A_1, A_2, \ldots A_i$, by i equations, giving the latter quantities in terms of the former.

Let now $i=\infty$; then each of the quantities ${}^ie_1$, ${}^ie_2$, ... ${}^ie_i$, will vanish, and it will follow that any series,

$$A_1 \sin\frac{1}{2}x + A_2 \sin\frac{3}{2}x + \&c.,$$

may be represented by a series of the form

$$a_1 \sin\left(\frac{1}{2}-c_1\right)x + a_2 \sin\left(\frac{3}{2}-c_2\right)x + \&c.$$

Now any function, fx, can be represented, between the limits $x=0$ and $x=\pi$, by the former series, and consequently by the latter also, between the same limits. But the latter series is equal to

$$a_1 \sin\frac{\varepsilon_1 x}{\pi} + a_2 \sin\frac{\varepsilon_2 x}{\pi} + \&c.;$$

and hence fx can be represented, between the limits 0 and π, by this series; and therefore it follows, that any function, Fx, can be represented, between the limits 0 and X, by the series

$$a_1 \sin n_1 x + a_2 \sin n_2 x + \&c.$$

P. Q. R.

IV.—NOTE ON A SOLUTION OF A CUBIC EQUATION.

By J. Cockle, B.A.

To the Editor of the Cambridge Mathematical Journal.

Sir,—Permit me to add a few remarks to the Solution of a Cubic Equation, which you did me the honour of inserting in your twelfth number.

Let α', β', γ', be the three roots of the original equation $f(x)=0$; then the roots of the transformed equation $f(y)=0$, will be $\alpha'-z$, $\beta'-z$, $\gamma'-z$, which let equal α, β, γ, respectively. By the known law of formation of the coefficients of an equation, the relation, among the coefficients of $f(y)=0$, $B^2=3.AC$ is the same as

$$(\alpha\beta+\alpha\gamma+\beta\gamma)^2=3\,.\,\alpha\beta\gamma\,.\,(\alpha+\beta+\gamma);$$

which, by expanding, transposing, and dividing by α^2, β^2, γ^2, becomes

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma}\,\ldots\ldots\,(1).$$

Solving this as a quadratic in $\frac{1}{\alpha}$, we find that part of the value of $\frac{1}{\alpha}$, which is under the radical sign, to be

$$-\frac{3}{4}\,.\left(\frac{1}{\beta}\sim\frac{1}{\gamma}\right)^2,$$

which, being essentially negative if β and γ are real (unless $\beta=\gamma$), shews that the equation (1) is inconsistent with the reality of *all* the quantities α, β, γ; if therefore α', β', γ', be all possible, z will be impossible, and the expression for x will assume an impossible form, unless two of its roots be equal; but when two of the roots of $f(x)=0$ are impossible, (1) may be satisfied by a possible value of z, and the possible root at once exhibited, as in the examples.

On solving the equation in z, that part of its value which is under the radical sign is

$$\frac{(ab-qc)^2-4\,(a^2-3b)\,(b^2-3ac)}{4\,(a^2-3b)^2};$$

and, the denominator being a complete square, the condition for z being possible is that the numerator should be not less than zero, or (reducing) that

$$81c^2+12b^3+12a^3c-3a^2b^2-54abc \mathrel{\overline{\overline{>}}} 0,$$

or, dividing by $3 \times 4 \times 27$,

$$\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3 + c\left(\frac{a}{3}\right)^3 - \frac{ab}{4 \times 27}(ab + 18c) \mathrel{\overline{\overline{>}}} 0,$$

which is the condition for z being possible, that is, of there being one and only one real root to the equation; or of all the roots being real and two equal; which last will be the case if the left-hand side of the above equals 0. If the above does not hold, all the roots are real and unequal.

Although there are two values of z, and three of the radical, which enters into the expression for x, yet these give only three values for x; since, obviously, the two values of z admit of no combinations with one another, but whichever value we take is used as if the other had no existence. It is indifferent which value is selected; thus, in the second example given of the method, the value $z = \frac{10}{4}$ gives the same result as $z = 1$.

Trin. Coll. Camb., May 19, 1841.

V.—ON THE INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS.

By B. Bronwin.

Let $(1 - x^2)\frac{d^2y}{dx^2} + my = 0$, $m = p(p-1)$, m and p integers

If $y = \Sigma a_n x^n$, we have for the determination of a_n the equation

$$n(n-1)a_n = \{(n-2)(n-3) - m\}\, a_{n-2}.$$

I shall call this the scale of the equation for the sake of convenience.

Making $n = 0$, 2, 4, &c. we have $a_{-2} = 0$, $a_{-4} = 0$, &c. and

$$a_2 = -\frac{m}{2}a_0,\ a_4 = \frac{m(m-1.2)}{2.3.4}a_0,\ a_6 = -\frac{m(m-1.2)(m-3.4)}{2.3.4.5.6}a_0, \text{ \&c.}$$

and making $n = 1$, 3, &c., $a_{-1} = 0$, $a_{-3} = 0$, &c.

$$a_3 = -\frac{m}{2.3}a_1,\ a_5 = \frac{m(m-2.3)}{2.3.4.5}a_1,\ a_7 = -\frac{m(m-2.3)(m-4.5)}{2.3.4.5.6.7}a_1, \text{ \&c.}$$

Consequently

$$y = a_0\left\{1 - \frac{m}{2}x^2 + \frac{m(m-1.2)}{2.3.4}x^4 - \text{\&c.}\right\} + a_1\left\{x - \frac{m}{2.3}x^3 + \frac{m(m-2.3)}{2.3.4.5}x^5 - \text{\&c.}\right\}$$

This is the complete integral of the proposed. One of the series which it contains will always terminate in the case supposed, and give a particular integral in finite terms, by means of which the other particular integral may be found in finite terms also.

Let $m=p=2$; we have $y=a_0(1-x^2)=C(1-x^2)$, a particular integral. Make $y=(1-x^2)z$. Putting this value in the proposed, we find

$$z=C\int\frac{dx}{(1-x^2)^2}=\tfrac{1}{4}C\left(\frac{2x}{1-x^2}+\log\frac{1+x}{1-x}\right)+C',$$

$$\text{and } y=C''\left\{2x+(1-x^2)\log\frac{1+x}{1-x}\right\}+C'(1-x^2),$$

for the complete integral.

Again, let $p=3$, or $m=6$; then $y=Cx(1-x^2)$, a particular integral. If $y=x(1-x^2)z$, we find

$$z=C\int\frac{dx}{x^2(1-x^2)^2}=\tfrac{3}{4}C\left\{\frac{2x^2-\frac{4}{3}}{x(1-x^2)}+\log\frac{1+x}{1-x}\right\}+C',$$

$$\text{and } y=C''\left\{2x^2-\tfrac{4}{3}+x(1-x^2)\log\frac{1+x}{1-x}\right\}+C'x(1-x^2),$$

for the complete integral.

In the general case the second particular integral may be found in finite terms, thus:

$$\text{Make } y=u+v\log\frac{1+x}{1-x},$$

where v is the particular integral which terminates. This value substituted in the proposed, gives

$$(1-x^2)\frac{d^2u}{dx^2}+mu+4\frac{dv}{dx}+\frac{4x}{1-x^2}v=0;$$

or if $v=(1-x^2)w$,

$$(1-x^2)\frac{d^2u}{dx^2}+mu+4(1-x^2)\frac{dw}{dx}-4xw=0.$$

If $u=\Sigma a_n x^n$, $w=\Sigma b_n x^n$, the scale of this is

$$n(n-1)a_n+\{m-(n-2)(n-3)\}a_{n-2}+4(n-1)b_{n-1}-4(n-2)b_{n-3}=0.$$

We now want the value of w. To obtain it we make $y=(1-x^2)z$. This value substituted in the proposed gives

$$(1-x^2)\frac{d^2z}{dx^2}-4x\frac{dz}{dx}+(m-2)z=0,$$

of which the scale is

$$n(n-1)\,b_n = \{n(n-1) - m\}\,b_{n-2},$$

which gives

$$z = b_0\left\{1 - \frac{m - 1.2}{2}x^2 + \frac{(m-1.2)(m-3.4)}{2.3.4}x^4 - \&c.\right\}$$

$$+\, b_1\left\{x - \frac{m-2.3}{2.3}x^3 + \frac{(m-2.3)(m-4.5)}{2.3.4.5}x^5 - \&c.\right\}.$$

The particular value of this which terminates gives w.

Returning now to the former equation, all the values of b_{-1}, b_{-2}, &c. and of a_{-1}, a_{-2}, &c. are nothing. And since we only want one arbitrary, which will be b_0 or b_1, we shall have a_0 or a_1 to assume at pleasure, but we shall not know how to assume it so as to make the series terminate. We shall therefore begin at the further extremity of the series, where $b_p = 0$, $b_{p+2} = 0$, &c.

If we make $a_{p+1} = 0$, we shall have a_{p+3}, a_{p+5}, &c. $= 0$, and we shall determine a_{p-1}, a_{p-3}, &c. in a retrograde order, and we shall have the value of u expressed by a series which terminates.

But we have z, or rather a particular value of z, by a descending series. Thus,

$$z = a_{p-2}\left\{x^{p-2} - \frac{(p-2)(p-3)}{2(2p-3)}x^{p-4} + \&c.\right\}.$$

Taking this for w, and making $a_{p+1} = 0$, we shall easily find u by a descending series which terminates.

If we integrate the proposed term by term successively, we have

$$(1-x^2)\frac{d^2y_1}{dx^2} + 2x\frac{dy_1}{dx} + (m-2)\,y_1 = 0, \quad y_1 = \int y\,dx;$$

$$(1-x^2)\frac{d^2y_2}{dx^2} + 4x\frac{dy_2}{dx} + (m-2.3)\,y_2 = 0, \quad y_2 = \int y_1\,dx;\ \&c.$$

$$(1-x^2)\frac{d^2y_{p-1}}{dx^2} + 2(p-1)\,x\frac{dy_{p-1}}{dx} = 0,$$

which is integrable, and will give a particular integral of the proposed.

If now we differentiate successively, we find

$$(1-x^2)\frac{d^2y_1}{dx^2} - 2x\frac{dy_1}{dx} + my_1 = 0, \quad y_1 = \frac{dy}{dx};$$

$$(1-x^2)\frac{d^2y_2}{dx^2}-4x\frac{dy_2}{dx}+(m-2)y_2=0,\quad y_2=\frac{dy_1}{dx};$$

$$(1-x^2)\frac{d^2y_3}{dx^2}-6x\frac{dy_3}{dx}+(m-2.3)y_3=0,\quad y_3=\frac{dy_2}{dx};\ \&c.$$

$$(1-x^2)\frac{d^2y_p}{dx^2}-2px\frac{dy_p}{dx}=0.$$

This will give the other particular integral of the proposed: and it is obvious that the integrals of all the preceding will easily be derived from the integral of the proposed.

We now proceed to a second example.

Let $(1+x^2)\dfrac{d^2y}{dx^2}-my=0$, m as before. The scale of this is

$$n(n-1)\,a_n=\{m-(n-2)(n-3)\}\,a_{n-2},$$

which gives

$$y=a_0\left\{1+\frac{m}{2}x^2+\frac{m(m-1.2)}{2.3.4}x^4+\&c.\right\}+a_1\left\{x+\frac{m}{2.3}x^3+\frac{m(m-2.3)}{2.3.4.5}x^5+\&c.\right\}$$

One of these series will terminate and give a particular integral in finite terms. Let $p=m=2$, then $y=C(1+x^2)$, a particular integral. Make $y=(1+x^2)z$, and we find by substitution in the proposed,

$$z=C\int\frac{dx}{(1+x^2)^2}=\tfrac{1}{2}C\left\{\frac{x}{1+x^2}+\tan^{-1}x\right\}+C',$$

and $y=C''\{x+(1+x^2)\tan^{-1}x\}+C'(1+x^2)$,

for the complete integral.

Again, let $p=3$, or $m=6$; then $y=Cx(1+x^2)$, a particular integral. And if we make $y=x(1+x^2)z$, we find

$$z=C\int\frac{dx}{x^2(1+x^2)^2}=-\tfrac{3}{2}C\left\{\frac{x^2+\frac{2}{3}}{x(1+x^2)}+\tan^{-1}x\right\}+C',$$

and $y=C''\{x^2+\frac{2}{3}+x(1+x^2)\tan^{-1}x\}+C'x(1+x^2)$,

for the complete integral.

By proceeding as in the last example, putting $\tan^{-1}x$ instead of $\log\dfrac{1+x}{1-x}$, we shall find the second particular integral in the general case in finite terms. Also,

$$(1+x^2)\frac{d^2y_1}{dx^2}-2x\frac{dy_1}{dx}-(m-2)y_1=0,\quad y_1=\int y\,dx;$$

$$(1+x^2)\frac{d^2y_2}{dx^2}-4x\frac{dy_2}{dx}-(m-2.3)y_2=0,\quad y_2=\int y_1\,dx;\ \&c.$$

$$(1+x^2)\frac{d^2y_1}{dx^2}+2x\frac{dy_1}{dx}-my_1=0,\quad y_1=\frac{dy}{dx},$$

$$(1+x^2)\frac{d^2y_2}{dx^2}+4x\frac{dy_2}{dx}-(m-2)\,y_2=0,\quad y_2=\frac{dy_1}{dx};\ \&c.$$

As a third example, suppose

$$(1-x^2)\frac{d^2y}{dx^2}+mx\frac{dy}{dx}-ry=0,\quad m=p+q-1,\quad r=pq.$$

The scale is $n(n-1)\,a_n=\{(n-2)(n-3)-m(n-2)+r\}\,a_{n-2}$,
or $n(n-1)\,a_n=(n-p-2)(n-q-2)\,a_{n-2}$.

This gives

$$y=a_0\left\{1+\frac{pq}{2}x^2+\frac{p(p-2)\,q\,(q-2)}{2\,.\,3\,.\,4}x^4+\&c.\right\}$$

$$+\,a_1\left\{x+\frac{(p-1)(q-1)}{2\,.\,3}x^3+\frac{(p-1)(p-3)(q-1)(q-3)}{2\,.\,3\,.\,4\,.\,5}x^5+\&c.\right\}$$

If one of the quantities p, q, be an even and the other an odd integer, both these series will terminate, and we shall have the complete integral in finite terms. Let $p=2$, $q=1$; $y=C(1+x^2)+C'x$ is the complete integral of

$$(1-x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-2y=0.$$

If one only of the quantities p, q be an integer, even or odd, we shall have a particular integral in finite terms; and we can express the other particular integral in finite terms by means of ordinary integrals.

Differentiating the proposed, making $y_1=\dfrac{dy}{dx}$, we have

$$(1-x^2)\frac{d^2y_1}{dx^2}+m'x\frac{dy_1}{dx}-r'y=0,$$

$$m'=m-2=p+q-3,\quad r'=r-m=(p-1)(q-1).$$

Hence p and q, by this operation, are each diminished of a unit. If therefore either of them be integer, we can, by repeating the operation, take away the last term from the equation, and render it integrable; but we should obtain only a particular integral.

For a fourth example, let

$$(1+x^2)\frac{d^2y}{dx^2}-mx\frac{dy}{dx}+ry=0,$$

m and r as before.

The scale of this is

$$n(n-1)a_n + (n-p-2)(n-q-2)a_{n-2} = 0;$$

whence we find

$$y = a_0 \left\{1 - \frac{pq}{2}x^2 + \frac{p(p-2)q(q-2)}{2.3.4}x^4 - \&c.\right\}$$

$$+ a_1 \left\{x - \frac{(p-1)(q-1)}{2.3}x^3 + \frac{(p-1)(p-3)(q-1)(q-3)}{2.3.4.5}x^5 - \&c.\right\}$$

If p and q be an even and an odd integer, both these series terminate. Let $p=2$, $q=1$, and we have $y = C(1-x^2) + C'x$ for the complete integral of

$$(1+x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

If one only of the quantities p, q be integer, we find only a particular integral in finite terms.

Differentiating the proposed once, making $\frac{dy}{dx} = y_1$, we obtain a similar equation, p and q being each diminished by unity. By successive differentiation it integrates itself, as in the last example, p or q being an integer.

Having obtained one particular integral, the other may be sometimes found by a simple transformation, as in the two next examples.

Example fifth.

$$\text{Let } x^2\frac{d^2y}{dx^2} - m\frac{dy}{dx} - ry = 0, \quad r = p(p-1).$$

The scale $mna_n = \{(n-1)(n-2) - r\}a_{n-1}$, gives immediately

$$y = a_0 \left\{1 - \frac{r}{m}x + \frac{r^2}{2m^2}x^2 - \frac{r^2(r-1.2)}{2.3m^3}x^3 + \frac{r^2(r-1.2)(r-2.3)}{2.3.4m^4}x^4 - \&c.\right\}$$

This is a particular integral, and it terminates in the case supposed. If v denote this integral, we find $y = Cv\int\frac{dx\, c^{-\frac{m}{x}}}{v^2}$ for the other particular integral, which we might suppose could not be freed from the sign of integration. But if we make $y = zc^{-\frac{m}{x}}$, the proposed becomes

$$x^2\frac{d^2z}{dx^2} + m\frac{dz}{dx} - \left(r + \frac{2m}{x}\right)z = 0.$$

The scale is $n(n-1)\,a_{n+1} + \{n(n-1) - r\}\,a_n = 0,$

or $n(n-1)\,a_{n+1} + (n-p)(n+p-1)\,a_n = 0,$

which gives

$$z = a_2\left\{x^2 + \frac{(p+1)(p-2)}{m}x^3 + \frac{(p+1)(p+2)(p-2)(p-3)}{2m^2}x^4 + \&c.\right\}$$

This terminates; and it is obvious that it is not the particular integral before found, since it cannot by any means be reduced to it: we have therefore the complete integral of the proposed in finite terms. And multiplying it by $c^{\frac{m}{x}}$, we have that of the equation

$$x^2\frac{d^2z}{dx^2} + m\frac{dz}{dx} - \left(r + \frac{2m}{x}\right)z = 0.$$

We can easily deduce also the integrals of the following:

$$x^2\frac{d^2y_1}{dx^2} + (2x - m)\frac{dy_1}{dx} - ry_1 = 0, \qquad y_1 = \frac{dy}{dx},$$

$$x^2\frac{d^2y_2}{dx^2} + (4x - m)\frac{dy_2}{dx} - (r-2)\,y_2 = 0, \qquad y_2 = \frac{dy_1}{dx},$$

$$x^2\frac{d^2y_3}{dx^2} + (6x - m)\frac{dy_3}{dx} - (r - 2\,.\,3)\,y_3 = 0, \quad y_3 = \frac{dy_2}{dx}, \ \&c.$$

$$x^2\frac{d^2y_1}{dx^2} - (2x + m)\frac{dy_1}{dx} - (r-2)\,y_1 = 0, \qquad y_1 = \int y\,dx,$$

$$x^2\frac{d^2y_2}{dx^2} - (4x + m)\frac{dy_2}{dx} - (r - 2\,.\,3)\,y_2 = 0, \quad y_2 = \int y_1\,dx, \ \&c.$$

Continuing these processes, the last term will ultimately vanish, and the equation become integrable.

Example sixth.

$$\text{Let } \frac{d^2y}{dx^2} + q\frac{dy}{dx} = \frac{m}{x^2}y, \quad m = p(p-1).$$

$$\text{Here } \{m - n(n-1)\}\,a_n = (n-1)\,qa_{n-1}.$$

Hence we find

$$y = a_0\left\{1 - \frac{m}{qx} + \frac{m(m-1\,.\,2)}{2q^2x^2} - \frac{m(m-1\,.\,2)(m-2\,.\,3)}{2\,.\,3\,q^3x^3} + \&c.\right\}$$

a particular integral, which terminates.

Make $y = zc^{-qx}$, and we have the transformed

$$\frac{d^2z}{dx^2} - q\frac{dz}{dx} = \frac{m}{x^2}z.$$

Hence we have only to change q into $-q$ in the value of y just found, and we have

$$z = a_0 \left\{1 + \frac{m}{qx} + \frac{m(m-1.2)}{2q^2x^2} + \frac{m(m-1.2)(m-2.3)}{2.3\,q^3x^3} + \&c.\right\}$$

therefore

$$y = C\left\{1 - \frac{m}{qx} + \frac{m(m-1.2)}{2q^2x^2} - \&c.\right\} + C'c^{-qx}\left\{1 + \frac{m}{qx} + \frac{m(m-1.2)}{2q^2x^2} + \&c.\right\}$$

is the complete integral of the proposed.

If in the value of y we change q into $-q$, we have the complete integral of

$$\frac{d^2y}{dx^2} - q\frac{dy}{dx} = \frac{m}{x^2}y.$$

Make $y = x^p u$, and we find by substitution

$$x\frac{d^2u}{dx^2} + (2p + qx)\frac{du}{dx} + qpu = 0.$$

The integral of this last, therefore, is immediately derivable from that of the proposed equation.

Again, make $\dfrac{dy}{dx} + qy = w$, and we have

$$x^2\frac{dw}{dx} = my, \quad x^2\frac{d^2w}{dx^2} + 2x\frac{dw}{dx} = m\frac{dy}{dx}.$$

Add to this the preceding multiplied by q, and we get

$$x^2\frac{d^2w}{dx^2} + (2x + qx^2)\frac{dw}{dx} = mw.$$

We have therefore the integral of this last also.

Make $y = vc^{-\frac{1}{2}qx}$, and the proposed becomes

$$\frac{d^2v}{dx^2} - \tfrac{1}{4}q^2v = \frac{m}{x^2}v;$$

of which we have also the integral. And changing q into $2q\sqrt{-1}$ in the integral of this last, we have that of

$$\frac{d^2v}{dx^2} + q^2v = \frac{m}{x^2}v.$$

But the integrals of the two last are better found directly; and the operation will show us, that when we cannot immediately obtain one particular integral, we may sometimes by a simple transformation find them both.

$$\text{Let } \frac{d^2y}{dx^2} - q^2y = \frac{m}{x^2}y, \quad m = p(p-1).$$

We cannot here obtain an integral immediately.

$$\text{Make } y = zc^{-qx}, \text{ then } \frac{d^2z}{dx^2} - 2q\frac{dz}{dx} = \frac{m}{x^2}z.$$

The scale of this is

$$\{n(n-1)-m\}\,a_n = (n-1)\,2qa_{n-1};$$

which gives

$$z = a_0\left\{1 + \frac{m}{2qx} + \frac{m(m-1\,.\,2)}{2(2qx)^2} + \&c.\right\}$$

$$\text{and } y = Cc^{-qx}\left\{1 + \frac{m}{2qx} + \frac{m(m-1\,.\,2)}{2(2qx)^2} + \&c.\right\}$$

If we change q into $-q$, we have

$$y = C'c^{qx}\left\{1 - \frac{m}{2qx} + \frac{m(m-1\,.\,2)}{2(2qx)^2} - \&c.\right\}$$

These are two particular integrals of the proposed, and give for the complete integral

$$y = Cc^{-qx}\left\{1 + \frac{m}{2qx} + \frac{m(m-1.2)}{2(2qx)^2} + \&c.\right\} + C'c^{qx}\left\{1 - \frac{m}{2qx} + \frac{m(m-1.2)}{2(2qx)^2} - \&c.\right\}$$

Changing q into $q\sqrt{-1}$, and suitably changing the arbitraries C, C', we shall easily find, by taking the odd and even terms separately,

$$y = C\sin(qx+\beta)\,.\left\{1 - \frac{m(m-1.2)}{1.2\,(2qx)^2} + \frac{m(m-1.2)(m-2.3)(m-3.4)}{1.2.3.4\,(2qx)^4} - \&c.\right\}$$

$$+\ C'\cos(qx+\beta)\,.\left\{\frac{m}{2qx} - \frac{m(m-1\,.\,2)(m-2\,.\,3)}{1\,.\,2\,.\,3\,(2qx)^3} + \&c.\right\}$$

for the complete integral of

$$\frac{d^2y}{dx^2} + q^2y = \frac{m}{x^2}y.$$

We might have ascending series instead of descending ones; but it would only be beginning at the other end of the series. And it may be observed, that the method employed in the last example is one that has been long applied to the integration of kindred equations.

In the last equation make $y = x^rz$, and it becomes

$$\frac{d^2z}{dx^2} + \frac{2r}{x}\frac{dz}{dx} + q^2z = 0, \quad r(r-1) = m.$$

Again, make $y = x^{-r}z$, and we have

$$\frac{d^2z}{dx^2} - \frac{2r}{x}\frac{dz}{dx} + q^2z = 0, \quad r(r+1) = m.$$

Thus the integrals of these two last are immediately obtained from the preceding value of y.

When we cannot obtain a complete integral, it will sometimes happen that a particular integral will suffice: and sometimes by means of ordinary integrals we can, with a little tact, obtain another in a simple form.

Let $\dfrac{d^2y}{dx^2} + q^2x\dfrac{dy}{dx} + q^2my = 0$, m an affirmative integer.

Make $y = zc^{-\frac{1}{2}q^2x^2}$.

This gives $\dfrac{d^2z}{dx^2} - q^2x\dfrac{dz}{dx} + q^2mz = 0$,

of which the scale is $n(n-1)a_n = (n-m-2)q^2a_{n-2}$.

Hence we find

$$z = C\left\{1 - \frac{m}{2}q^2x^2 + \frac{m(m-2)}{2.3.4}q^4x^4 - \frac{m(m-2)(m-4)}{2.3.4.5.6}q^6x^6 + \&c.\right\}$$

$$+ C'\left\{qx - \frac{m-1}{2.3}q^3x^3 + \frac{(m-1)(m-3)}{2.3.4.5}q^5x^5 - \&c.\right\}$$

and therefore

$$y = Cc^{-\frac{1}{2}q^2x^2}\left\{1 - \frac{m}{2}q^2x^2 + \frac{m(m-2)}{2.3.4}q^4x^4 - \&c.\right\} + C'c^{-\frac{1}{2}q^2x^2}\left\{qx - \frac{m-1}{2.3}q^3x^3 + \&c.\right\}$$

In each of these integrals one of the series which they contain will terminate; the first if m be even, the second if odd.

If we change q^2 into $-q^2$, and suitably change the arbitraries, we have

$$\frac{d^2y}{dx^2} - q^2x\frac{dy}{dx} - q^2my = 0, \quad \frac{d^2z}{dx^2} + q^2x\frac{dz}{dx} - q^2mz = 0,$$

$$z = C\left\{1 + \frac{m}{2}q^2x^2 + \frac{m(m-2)}{2.3.4}q^4x^4 + \&c.\right\} + C'\left\{qx + \frac{m-1}{2.3}q^3x^3 + \&c.\right\}$$

$$y = Cc^{\frac{1}{2}q^2x^2}\left\{1 + \frac{m}{2}q^2x^2 + \frac{m(m-2)}{2.3.4}q^4x^4 + \&c.\right\} + C'c^{\frac{1}{2}q^2x^2}\left\{qx + \frac{m-1}{2.3}q^3x^3 + \&c.\right\}$$

Thus we have, in finite terms, a particular integral of each of the four preceding equations. Before we proceed to find the other particular integral, it will be convenient to have that already found in a descending series.

If $\dfrac{d^2y}{dx^2} + q^2x\dfrac{dy}{dx} - q^2my$; $n(n-1)a_n = (m-n+2)q^2a_{n-2}$, and

$$y = Cx^m \left\{1 + \frac{m(m-1)}{2q^2x^2} + \frac{m(m-1)(m-2)(m-3)}{2.4q^4x^4} + \&c.\right\},$$

which terminates whether m be odd or even.

Changing q^2 into $-q^2$, we have

$$\frac{d^2y}{dx^2} - q^2x\frac{dy}{dx} + q^2my = 0,$$

and $y = Cx^m \left\{1 - \frac{m(m-1)}{2q^2x^2} + \frac{m(m-1)(m-2)(m-3)}{2.4q^4x^4} - \&c.\right\}$:

and from what has been done,

$$\frac{d^2y}{dx^2} + q^2x\frac{dy}{dx} + q^2my = 0;$$

$$y = Cx^m c^{-\frac{1}{2}q^2x^2} \left\{1 - \frac{m(m-1)}{2q^2x^2} + \frac{m(m-1)(m-2)(m-3)}{2.4q^4x^4} - \&c.\right\}:$$

$$\frac{d^2y}{dx^2} - q^2x\frac{dy}{dx} - q^2my = 0;$$

$$y = Cx^m c^{\frac{1}{2}q^2x^2} \left\{1 + \frac{m(m-1)}{2q^2x^2} + \frac{m(m-1)(m-2)(m-3)}{2.4q^4x^4} + \&c.\right\}.$$

We are now prepared to find the other particular integral. For this purpose we will take

$$\frac{d^2z}{dx^2} - q^2x\frac{dz}{dx} + q^2mz = 0.$$

$$\text{Make } z = uc^{\frac{1}{2}q^2x^2} + v\int dx\, c^{\frac{1}{2}q^2x^2},$$

where v is the particular integral already found. Substituting this value, we have

$$\frac{d^2u}{dx^2} + q^2x\frac{du}{dx} + q^2mu + 2\frac{dv}{dx} = 0.$$

If $u = \Sigma a_n x^n$, $v = \Sigma b_n x^n$, the scale is

$$n(n-1)a_n + (n+m-2)q^2a_{n-2} + 2(n-1)b_{n-1} = 0,$$
$$n(n-1)b_n - (n-m-2)q^2b_{n-2} = 0,$$

Now b_{m+2}, b_{m+4}, &c. = 0. Make $a_{m+1} = 0$; then a_{m+3}, a_{m+5}, &c. = 0, and we have

$$a_{m-1} = -\frac{2m}{2m-1}\cdot\frac{b_m}{q^2}, \quad a_{m-3} = -\frac{m(m-1)(m-2)}{2m-1}\cdot\frac{b_m}{q^4}, \text{ \&c.}$$

The scale shows that the series terminates either at a_1 or a_0, according as m is an odd or an even number. And thus we shall have

$$u = a_{m-1}x^{m-1} + a_{m-3}x^{m-3} + \dots + a_1x, \text{ or } a_0.$$

Multiplying the particular integral thus found by $c^{-\frac{1}{2}q^2x^2}$, and then changing q^2 into $-q^2$, we shall have the other three particular integrals.

By successive differentiation and integration of the four equations last integrated, we should ultimately make the last term to vanish, and the equations to integrate themselves. But for the most part we obtain only particular integrals by such a process.

As another example, let $\dfrac{d^2y}{dx^2}+q\dfrac{dy}{dx}+\dfrac{mq}{x}y=0$, m a positive integer. Make $y=zc^{-qx}$, and we have the transformed

$$\frac{d^2z}{dx^2}-q\frac{dz}{dx}+\frac{mq}{x}z=0.$$

$$\text{Here } n(n-1)a_n=(n-m-1)qa_{n-1};$$

$$\text{whence } z=Cx\left\{1-\frac{m-1}{2}qx+\frac{(m-1)(m-2)}{2^2.3}q^2x^2-\&\text{c.}\right\}$$

$$y=Cxc^{-qx}\left\{1-\frac{m-1}{2}qx+\frac{(m-1)(m-2)}{2^2.3}q^2x^2-\&\text{c.}\right\}$$

Changing q into $-q$, we obtain particular integrals of

$$\frac{d^2y}{dx^2}-q\frac{dy}{dx}-\frac{mq}{x}y=0,\quad \frac{d^2z}{dx^2}+q\frac{dz}{dx}-\frac{mq}{x}z=0.$$

All these integrals terminate in the case supposed. Found by a descending series, they are

$$z=Cx^m\left\{1-\frac{m(m-1)}{qx}+\frac{m(m-1)^2(m-2)}{2q^2x^2}-\frac{m(m-1)^2(m-2)^2(m-3)}{2.3q^3x^3}+\&\text{c.}\right\}$$

$$y=Cx^mc^{-qx}\left\{1-\frac{m(m-1)}{qx}+\frac{m(m-1)^2(m-2)}{2q^2x^2}-\&\text{c.}\right\}$$

Changing q into $-q$, we obtain the corresponding integrals of the two last equations.

In order to have the other particular integral, we will take the equation

$$\frac{d^2z}{dx^2}-q\frac{dz}{dx}+\frac{qm}{x}z=0,$$

and we make

$$z=uc^{qx}+v\int\frac{dx}{x}c^{qx},$$

where v is the particular integral which we have found, whether by an ascending or a descending series. By substitution we find

$$\frac{d^2u}{dx^2} + q\frac{du}{dx} + \frac{qm}{x}u + \frac{2}{x}\frac{dv}{dx} - \frac{1}{x^2}v = 0.$$

If $u = \Sigma a_n x^n$, $v = \Sigma b_n x^n$, the scale is

$$n(n-1)\,a_n + (n+m-1)\,qa_{n-1} + (2n-1)\,b_n = 0,$$

$$\text{and } n(n-1)\,b_n = (n-m-1)\,qb_{n-1}.$$

As we only want a particular integral, we have one of the quantities a_0, a_1, a_2, &c. more than we want, since we have an arbitrary in the value of v. We may therefore make $a_m = 0$; then a_{m+1}, a_{m+2}, ... $= 0$; since $b_{m+1} = 0$, $b_{m+2} = 0$, &c.

We begin at the further extremity of the series, and find the quantities a_0, a_1, &c. in a retrograde order, and we may take either the ascending or descending series for v; but the latter is more convenient. Thus we have

$$a_{m-1} = -\frac{b_m}{q} = -\frac{C}{q},\quad a_{m-2} = \{m(m-1)-1\}\frac{C}{q^2},\ \&c.;$$

$$\text{also } a_0 = -\frac{b_1}{mq},\quad a_{-1} = 0;$$

$$\text{therefore } u = a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \dots + a_0.$$

From this, by multiplying by c^{-qx}, and changing q into $-q$ as before, we derive the other three particular integrals.

It is often very important to attend particularly to the scale; for it will often direct us to the right mode of finding the integral, when otherwise we might have overlooked it. We will show this in a few examples, though it must in a great measure have been apparent as we have proceeded.

$$\text{Let } \frac{d^2y}{dx^2} + \frac{m}{x}\frac{dy}{dx} + \frac{r}{x^2}y = 0.$$

The scale is $\{n(n-1) + mn + r\}\,a_n = 0$, or $n(n-1) + mn + r = 0$.

Let n_1, n_2, be the values of n which satisfy this equation; then

$$y = Cx^{n_1} + C'x^{n_2}.$$

It is true the equation here taken as an example is one which we already knew how to integrate, and one of the easiest to integrate, except where the coefficients are constant: but it is not integrated so easily in any other way.

$$\text{Next, let } \frac{d^2y}{dx^2} + q^2x\frac{dy}{dx} + q^2y = 0.$$

The scale of this is

$$n(n-1)\,a_n + (n-1)\,q^2 a_{n-2} = 0;$$

which reduces to $na_n + q^2 a_{n-2} = 0$.

This last is the scale of $\dfrac{dy}{dx} + q^2 xy = 0$; which gives $y = Cc^{-\frac{1}{2}q^2x^2}$, a particular integral of the proposed. The complete integral is

$$y = Cc^{-\frac{1}{2}q^2x^2}\int dx c^{\frac{1}{2}q^2x^2} + C'c^{-\frac{1}{2}q^2x^2}.$$

The proposed may be put under the form $\dfrac{d^2y}{dx^2} + q^2\,\dfrac{d(xy)}{dx} = 0$, and is immediately integrable; but we might have overlooked that circumstance if the scale had not pointed it out to us.

$$\text{Again, let } (1-x^2)\,\frac{d^2y}{dx^2} + 2y = 0.$$

The scale of this is

$$n(n-1)\,a_n = \{(n-2)(n-3) - 2\}\,a_{n-2} = (n-4)(n-1)\,a_{n-2},$$
$$\text{or } na_n = (n-4)\,a_{n-2}.$$

This is the scale of $(1-x^2)\,\dfrac{dy}{dx} + 2xy = 0$; which gives $y = C(1-x^2)$, a particular integral of the proposed. In fact, the proposed is immediately integrable, and gives

$$(1-x^2)\,\frac{dy}{dx} + 2xy = C.$$

These may be thought to be very simple examples, the integrability of which was easily discovered. But were I not afraid of extending this paper to too great a length, it would not be difficult to show the importance of consulting the scale in cases much less obvious.

VI.—ON THE MOTION OF A SPHERE PROJECTED ALONG A CYLINDER REVOLVING UNIFORMLY IN A VERTICAL PLANE.

By James Booth, M.A., Principal of and Professor of Mathematics in Bristol College.

Let a hollow cylinder be supposed to revolve uniformly in a vertical plane round a horizontal axe, and let a sphere be projected along the cylinder from a given point in it

with a given velocity; to determine the motion of the sphere, excluding the effects of friction and the resistance of the atmosphere.

Let r be the distance of the sphere from the centre of motion at the end of the time t; let T be the period of the revolution of the cylinder, a the angle which the axis of the cylinder makes with the horizontal axe in its own plane at the beginning of the time t; then the forces which act on the sphere at any instant, are the force of gravity resolved along the axis of the cylinder, and the centrifugal force arising from the uniform rotation of the cylinder.

Now the angle which the axis of the cylinder makes with the horizontal axe at the end of the time t, is $\left(a + 2\pi \frac{t}{T}\right)$; hence the force of gravity resolved along the axis of the cylinder is $-g \sin\left(a + \frac{2\pi t}{T}\right)$, g being taken with a negative sign, as it acts towards the centre at the commencement of the motion, and the centrifugal force is $\frac{4\pi^2 r}{T^2}$; hence the differential equation of the motion of the sphere is

$$\frac{d^2r}{dt^2} = \frac{4\pi^2 r}{T^2} - g \sin\left(a + 2\pi \frac{t}{T}\right) \ldots\ldots (1).$$

Let $\frac{2\pi}{T} = k$, and equation (1) becomes

$$\frac{d^2r}{dt^2} = k^2 r - g \sin(a + kt) \ldots\ldots\ldots (2).$$

Now, to integrate this equation, using the method of the variation of parameters, let us assume

$$r = Ae^{kt} + Be^{-kt} \ldots\ldots\ldots (3),$$

which is the integral of the differential equation $\frac{d^2r}{dt^2} = k^2 r$; A and B being functions of t to be determined.

Differentiating (3), we obtain

$$\frac{dr}{dt} = k(Ae^{kt} - Be^{-kt}) + \frac{dA}{dt} e^{kt} + \frac{dB}{dt} e^{-kt} \ldots\ldots (4).$$

As a first condition, let

$$\frac{dA}{dt} e^{kt} + \frac{dB}{dt} e^{-kt} = 0 \ldots\ldots (5).$$

Introducing the condition (5) into (4), and differentiating it again, we find

$$\frac{d^2r}{dt^2} = k^2(Ae^{kt} + Be^{-kt}) + k\left(\frac{dA}{dt}e^{kt} - \frac{dB}{dt}e^{-kt}\right).$$

Eliminating from this equation the quantity $Ae^{kt} + Be^{-kt}$ by (3), we get

$$\frac{d^2r}{dt^2} = k^2r + k\left(\frac{dA}{dt}e^{kt} - \frac{dB}{dt}e^{-kt}\right) \ldots\ldots (6).$$

Comparing this equation with (2), we find

$$\frac{dA}{dt}e^{kt} - \frac{dB}{dt}e^{-kt} = \frac{-g}{k}\sin(a + kt) \ldots\ldots (7).$$

Eliminating successively $\frac{dB}{dt}$, $\frac{dA}{dt}$, from equations (5) and (7), and integrating by parts, we obtain

$$\left.\begin{aligned} A &= C + \frac{g}{4k^2}e^{-kt}\{\sin(a+kt) + \cos(a+kt)\} \\ B &= C' + \frac{g}{4k^2}e^{kt}\{\sin(a+kt) - \cos(a+kt)\} \end{aligned}\right\} \ldots (8),$$

C and C' being arbitrary constants. Substituting these values of A and B in (3), we find

$$r = Ce^{kt} + C'e^{-kt} + \frac{g}{2k^2}\sin(a + kt) \ldots\ldots (9).$$

To determine the values of C and C'.

Let R be the initial distance, and V the velocity of projection; then R and V are the values of r and $\frac{dr}{dt}$ when $t = 0$; hence

$$\left.\begin{aligned} R &= C + C' + \frac{g}{2k^2}\sin a \\ \frac{V}{k} &= C - C' + \frac{g}{2k^2}\cos a \end{aligned}\right\} \ldots (10).$$

From these equations, determining the values of C and C', and substituting them in (9), we obtain finally the equation

$$r = \left(\frac{R}{2} - \frac{g}{4k^2}\sin a\right)(e^{kt} + e^{-kt})$$
$$+ \left(\frac{V}{2k} - \frac{g}{4k^2}\cos a\right)(e^{kt} - e^{-kt}) + \frac{g}{2k^2}\sin(a + kt) \ldots (11),$$

from which we obtain the value of r in terms of the time.

1. Let the initial distance $=\frac{g}{2k^2}\sin a$, and the initial velocity $=\frac{g}{2k}\cos a$; then equation (11) is changed into

$$r=\frac{g}{2k^2}\sin(a+kt)\ldots\ldots(12).$$

the equation of a circle, whose lowest point is in the centre of motion, and whose radius $=\frac{g}{4k^2}$.

In order to simplify, let the initial position of the cylinder be vertical, and the initial velocity $=0$; then $a=\frac{\pi}{2}$, $V=0$, and

$$r=\left(\frac{R}{2}-\frac{g}{4k^2}\right)(e^{kt}+e^{-kt})+\frac{g}{2k^2}\cos kt\ldots\ldots(13).$$

In the first place, if the initial distance be taken $=\frac{g}{2k^2}$, the orbit which the sphere describes is a vertical circle, touching at its lowest point the horizontal line passing through the centre of motion; and this circle is described by the sphere in a semi-revolution of the cylinder.

At the end of one or any odd number of revolutions, the sphere will be found at the negative side of the origin in the cylinder, at the distance R from the centre; while at the end of two or any even number of revolutions it returns to the same point of space and the same position in the cylinder.

2. When R is $\gtrless\frac{g}{2k^2}$, let $R-\frac{g}{2k^2}=2\delta$, and (13) becomes

$$r=\delta(e^{kt}+e^{-kt})+\frac{g}{2k^2}\cos kt\ldots\ldots(14).$$

In this case the sphere will alternately pass to and fro through the centre of motion so long as $\delta(e^{kt}+e^{-kt})$ is less than $\frac{g}{2k^2}$, making excursions in the negative arm of the cylinder, constantly diminishing, until at length it remains altogether on the positive side of the origin, receding with a constantly accelerated velocity from the centre of motion, ever approaching in the upper part of the orbit to the centre by a fixed quantity, and receding from it in the lower part by the same.

The sphere in this case undulates along a curve which is asymptotic to a logarithmic spiral, the angle between whose

tangent and rad. vec. is equal to half a right angle, and whose parameter is δ.

3. Let the sphere have no initial velocity, and let the initial position of the cylinder be horizontal; then $V = 0$, $a = 0$; putting these values in (11),

$$r = \left(\frac{R}{2} - \frac{g}{4k^2}\right) e^{kt} + \left(\frac{R}{2} + \frac{g}{4k^2}\right) e^{-kt} + \frac{g}{2k^2} \sin kt.$$

Let $R = \frac{g}{2k^2}$, then

$$r = \frac{g}{2k^2} (e^{-kt} + \sin kt);$$

in which case the orbit approaches indefinitely to a circle.

When the motion of the cylinder is very slow, k becomes very small, and (11) is changed into $r = R + \frac{R}{2} k^2 t^2 - \frac{g}{2} t^2$; which, when $k = 0$, gives $r = R - \frac{gt^2}{2}$, as it ought to be.

Let the period of revolution be $2''$; then $T = 2''$,

$$k = \frac{2\pi}{T} = \pi, \quad g = 32.1908 \text{ feet};$$

$$\text{hence } R = \frac{g}{2k^2} = \frac{g}{2\pi^2} = 1.630 \text{ feet},$$

the diameter of the vertical circle which the sphere describes.

VII.—ON THE DETERMINATION OF THE INTENSITY OF VIBRATION OF WAVELETS DIVERGING FROM EVERY POINT OF A PLANE WAVE.

Calculations of phenomena of diffraction on the principles of Fresnel, are made by supposing that from every part of the front of a primary wave, in any position, small waves diverge, proportional in intensity to the superficial extent of the part, and diminishing in intensity proportionally to the distance through which they diverge. So that if $a \cdot \sin \frac{2\pi}{\lambda}(vt - x)$ re-

present the disturbance of a particle in the front of the primary wave, from every element dS of the surface we may suppose a small wave to proceed, and the disturbance of a particle by the small wave to be represented by

$$\frac{b}{r} . \, dx \, dy \sin \frac{2\pi}{\lambda} . (vt - c - r).$$

The value of the coefficient b depends on that of a, but the relation is not, I believe, given by Fresnel nor by Airy, nor, as far as I know, by any other writer. I propose to determine it from the principle, that if the primary wave be plane, the disturbance caused by the small waves at any point in front of the primary wave must be the same as the disturbance which would be caused by the primary wave itself. Let c be the length of a perpendicular drawn from a point A to the front of a plane wave, r and u the distances of any element of the wave from the point A, and from the intersection of the perpendicular with the front of the primary wave, so that $r^2 = c^2 + u^2$. Supposing the front of the wave to be divided into elementary zones by concentric circles of radii u and $u + du$, the area of each is $2\pi u du = 2\pi r dr$. The disturbance produced at the point A by waves proceeding from this zone will be, since the waves proceeding from each point are in the same phase,

$$b . 2\pi \, dr \sin \frac{2\pi}{\lambda} . (vt - r),$$

and the total disturbance

$$= 2\pi b \int_c^\infty \sin \frac{2\pi}{\lambda} (vt - r) \, dr$$

$$= b\lambda \left\{ \cos \frac{2\pi}{\lambda} (vt - \infty) - \cos \frac{2\pi}{\lambda} (vt - c) \right\},$$

$$= b\lambda \sin \left\{ \frac{2\pi}{\lambda} . (vt - c) - \frac{\pi}{2} \right\},$$

(since the sines and cosines of infinite arcs are 0,)

$$= b\lambda \sin \frac{2\pi}{\lambda} \left\{ vt - \left(c + \frac{\lambda}{4} \right) \right\},$$

so that $b\lambda = a$, and therefore the coefficients of the small waves diverging from dS must be $\dfrac{adS}{r\lambda}$.

It would also appear from this, that we should consider the phase of the small wave to be expressed by

$$\sin\frac{2\pi}{\lambda}\left\{vt-\left(r-\frac{\lambda}{4}\right)\right\};$$

so that if the substitution of small waves for the primary wave were real instead of hypothetical, there would be a loss of a quarter of an undulation.

H. T.

VIII.—MATHEMATICAL NOTE.

A Suggestion in Notation.—Of all the repetitions which want of notation compels, that of n, $n\frac{n-1}{2}$, $n\frac{n-1}{2}\frac{n-2}{3}$, is one of the worst. If these were to be called n_1, n_2, n_3, &c. the notation could not be permanent, since n_1, n_2, &c. are used for any set of quantities following a law. But if 1_n, 2_n, 3_n, &c. be used, no existing notation will be interfered with, except in the *general term* k_n, which it is seldom wanted to pass over quickly and frequently.

These symbols, 1_n, 2_n, 3_n, &c. might be read '1 out of n', '2 out of n', &c. or '1 of n', '2 of n', &c. in abbreviation of 'the number of ways in which 1, 2, &c. may be selected out of n'. The following are some instances of their use:

$$(1+x)^n = 1 + 1_n x + 2_n x^2 + 3_n x^3 + 4_n x^4 + \ldots.$$

$$k_{m+n} = k_m + (k-1)_m 1_n + (k-2)_m 2_n + (k-3)_m 3_n + \ldots.$$

$$\frac{k_n}{l_n} = \frac{(k-l)_{n-l}}{(k-l)_k},\quad k_n = \frac{\Gamma(n)}{\Gamma(k)\,\Gamma(n-k)},\quad (k+1)_n = k_n \times \frac{n-k}{k+1}.$$

A. D. M.

ERRATUM.

Page 22, line 4, *for* a *read* $\frac{a_{m-2}}{a_{m-1}}$.

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I.—ON THE MOTION OF A PARTICLE ALONG VARIABLE AND MOVEABLE TUBES AND SURFACES.

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THE earliest problem of the motion of a particle along a moveable tube of invariable form, is one given by John Bernoulli, (*Opera*, tom. 4, p. 248,) where the tube is rectilinear, and is made to revolve in a horizontal plane about one extremity with a uniform angular velocity. A solution of this problem is given also by Clairaut, to whom it had probably been proposed by Bernoulli, in the *Mémoires de l'Académie des Sciences de Paris*, for the year 1742, p. 10. A similar problem, which had been erroneously attempted by Barbier in the *Annales de Gergonne*, tom. 19, was correctly solved in the following volume by Ampère: in this problem the tube is supposed to revolve uniformly in a vertical instead of a horizontal plane, about the fixed extremity, the particle being consequently subject to the action of gravity. In the last number of this Journal may be seen a solution of this problem, by Professor Booth, who has discussed at length the more interesting cases of the motion. Several problems of a like character are to be met with, in which the tube is of invariable form, and is made to revolve about a fixed point with a uniform angular velocity. The object of this paper is to give a general method for the determination of the motion of a particle within tubes, and between contiguous surfaces of which either the position, or the form, or both, are made to

vary according to any assigned law whatever, the particle being acted on by any given forces. We will commence with the consideration of the motion of a particle along a tube, and for the sake of perfect generality, we will suppose the tube to be one of double curvature. The tube is considered in all cases to be indefinitely narrow and perfectly smooth, and every section at right angles to its axis to be circular.

Let the particle be referred to three fixed rectangular axes, and let x, y, z be its co-ordinates at any time t; let x, y, z become $x + \delta x$, $y + \delta y$, $z + \delta z$, when t becomes $t + \delta t$; δt, and consequently $\delta x, \delta y, \delta z$ being considered to be indefinitely small. Then the effective accelerating forces on the particle parallel to the three fixed axes will be at the time t,

$$\frac{\delta^2 x}{\delta t^2}, \quad \frac{\delta^2 y}{\delta t^2}, \quad \frac{\delta^2 z}{\delta t^2}.$$

Also, let X, Y, Z, represent the impressed accelerating forces on the particle resolved parallel to the axes of x, y, z; and let $x + dx$, $y + dy$, $z + dz$, be the co-ordinates of a point in the tube very near to the point x, y, z, which the particle occupies at the time t. Then, observing that the action of the tube on the particle is always at right angles to its axis at every point and therefore at the time t to the line joining the two points x, y, z, and $x + dx$, $y + dy$, $z + dz$, we have, by D'Alembert's Principle, combined with the Principle of Virtual Velocities,

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right) dx + \left(\frac{\delta^2 y}{\delta t^2} - Y\right) dy + \left(\frac{\delta^2 z}{\delta t^2} - Z\right) dz = 0 \ldots. \text{ (I.)}.$$

Again, since the form and position of the tube are supposed to vary according to some assigned law, it is clear that when t is known the equations to the tube must be known; hence it is evident that in addition to the equation (I.) we shall have, from the particular conditions of each individual problem, a number of equations equivalent to two of the form

$$\phi(x, y, z, t) = 0, \quad \chi(x, y, z, t) = 0 \ldots. \text{ (II.)},$$

where ϕ and χ are symbols of functionality depending upon the law of the variations of the form and position of the tube.

The three equations (I.) and (II.) involve the four quantities x, y, z, t, and therefore in any particular case, if the difficulty of the analytical processes be not insuperable, we may ascertain x, y, z, each of them in terms of t; in which consists the complete solution of the problem.

If the tube remain during the whole of the motion within one plane; then the plane of x, y, being so chosen as to coincide with this plane, the three equations (I.) and (II.) will evidently be reduced to the two

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right) dx + \left(\frac{\delta^2 y}{\delta t^2} - Y\right) dy = 0 \;\ldots\ldots\; \text{(III.)},$$

$$\phi(x, y, t) = 0 \;\ldots\ldots\ldots\ldots\; \text{(IV.)}$$

We proceed to illustrate the general formulæ of the motion by the discussion of a few problems.

1. A rectilinear tube revolves with a uniform angular velocity about one extremity in a horizontal plane; to find the motion of a particle within the tube. This is Bernoulli's problem.

Let ω be the constant angular velocity; r the distance of the particle at any time t from the fixed extremity of the tube; then the plane of x, y, being taken horizontal, and the origin of co-ordinates at the fixed extremity of the tube, we shall have, supposing the tube initially to coincide with the axis of x,

$$x = r \cos \omega t \;\ldots\ldots\; (1),$$

$$y = r \sin \omega t \;\ldots\ldots\; (2).$$

From (1) we have

$$dx = dr \cos \omega t,$$

and from (2),

$$dy = dr \sin \omega t.$$

Again, from (1) we have

$$\frac{\delta x}{\delta t} = \frac{\delta r}{\delta t} \cos \omega t - \omega r \sin \omega t,$$

$$\frac{\delta^2 x}{\delta t^2} = \frac{\delta^2 r}{\delta t^2} \cos \omega t - 2\omega \frac{\delta r}{\delta t} \sin \omega t - \omega^2 r \cos \omega t;$$

and from (2),

$$\frac{\delta y}{\delta t} = \frac{\delta r}{\delta t} \sin \omega t + \omega r \cos \omega t,$$

$$\frac{\delta^2 y}{\delta t^2} = \frac{\delta^2 r}{\delta t^2} \sin \omega t + 2\omega \frac{\delta r}{\delta t} \cos \omega t - \omega^2 r \sin \omega t.$$

Substituting in the general formula (III.) the values which we have obtained for dx, dy, $\frac{\delta^2 x}{\delta t^2}$, $\frac{\delta^2 y}{\delta t^2}$, we have, since $X = 0$, $Y = 0$,

$$\cos \omega t \left(\frac{\delta^2 r}{\delta t^2} \cos \omega t - 2\omega \frac{\delta r}{\delta t} \sin \omega t - \omega^2 r \cos \omega t\right)$$

$$+ \sin \omega t \left(\frac{\delta^2 r}{\delta t^2} \sin \omega t + 2\omega \frac{\delta r}{\delta t} \cos \omega t - \omega^2 r \sin \omega t\right) = 0;$$

and therefore

$$\frac{\delta^2 r}{\delta t^2} - \omega^2 r = 0:$$

the integral of this equation is

$$r = C\epsilon^{\omega t} + C'\epsilon^{-\omega t}.$$

Let $r = a$ when $t = 0$; then

$$a = C + C';$$

also let $\frac{\delta r}{\delta t} = \beta$ when $t = 0$; then

$$\beta = C\omega - C'\omega;$$

from the two equations for the determination of C and C', we have

$$C = \frac{a\omega + \beta}{2\omega}, \quad C' = \frac{a\omega - \beta}{2\omega};$$

hence for the motion of the particle along the tube

$$2\omega r = (a\omega + \beta)\,\epsilon^{\omega t} + (a\omega - \beta)\,\epsilon^{-\omega t}.$$

2. In the case of Ampère's problem, we shall have by the same process, the axis of y being now taken vertical, observing that $X = 0$, $Y = -g \sin \omega t$, if the time be reckoned from the moment of coincidence of the tube with the axis of x which is horizontal,

$$\frac{\delta^2 r}{\delta t^2} - \omega^2 r = -g \sin \omega t.$$

The integral of this equation is

$$r = C\epsilon^{\omega t} + C'\epsilon^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t;$$

and if we determine the constants from the conditions that r, $\frac{dr}{dt}$ shall have initially values a, β, we shall get for the motion along the tube,

$$2\omega r = \left(a\omega + \beta - \frac{g}{2\omega}\right)\epsilon^{\omega t} + \left(a\omega - \beta + \frac{g}{2\omega}\right)\epsilon^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t.$$

3. Suppose the tube to revolve in a horizontal plane about a fixed extremity with such an angular velocity, that the tangent of its angle of inclination to the axis of x is proportional to the time.

The equation to the tube at any time t will be

$$y = mtx \dots\dots (1),$$

where m is some constant quantity; hence

$$dy = mt\, dx,$$

and therefore from (III.), since $X = 0$ and $Y = 0$,

$$\frac{\delta^2 x}{\delta t^2} + mt \frac{\delta^2 y}{\delta t^2} = 0 \dots\dots (2).$$

But from (1) we have

$$\frac{\delta y}{\delta t} = mt \frac{\delta x}{\delta t} + mx,$$

$$\frac{\delta^2 y}{\delta t^2} = mt \frac{\delta^2 x}{\delta t^2} + 2m \frac{\delta x}{\delta t};$$

hence from (2),

$$(1 + m^2 t^2) \frac{\delta^2 x}{\delta t^2} + 2m^2 t \frac{\delta x}{\delta t} = 0,$$

$$\frac{\frac{\delta^2 x}{\delta t^2}}{\frac{\delta x}{\delta t}} + \frac{2m^2 t}{1 + m^2 t^2} = 0.$$

Integrating, we have

$$\log \frac{\delta x}{\delta t} + \log (1 + m^2 t^2) = \log C,$$

$$\frac{\delta x}{\delta t} (1 + m^2 t^2) = C.$$

Let β be the initial value of $\dfrac{\delta x}{\delta t}$, which will be the velocity of projection along the tube; then $C = \beta$, and therefore

$$\frac{\delta x}{\delta t} (1 + m^2 t^2) = \beta,\ \delta x = \frac{\beta \delta t}{1 + m^2 t^2};$$

integrating, we get

$$x + C = \frac{\beta}{m} \tan^{-1} (mt).$$

Let $x = a$ when $t = 0$; then $a + C = 0$, and therefore

$$x = a + \frac{\beta}{m} \tan^{-1} (mt),$$

and consequently from (1)

$$y = amt + \beta t \tan^{-1} (mt).$$

If θ be the inclination of the tube to the axis of x at any time, and r be the distance of the particle from the fixed extremity,

$$r = \frac{am + \beta\theta}{m \cos \theta}.$$

4. A circular tube is constrained to move in a horizontal plane with a uniform angular velocity about a fixed point in its circumference; to determine the motion of a particle within the tube, which is placed initially at the extremity of the diameter passing through the fixed point.

Let the fixed point be taken as the origin of co-ordinates, and let the axis of x coincide with the initial position of the diameter through this point; let ω be the angular velocity of the revolution of the circle, a the radius; also let θ be the angle at any time t between the distance of the particle and of the extremity of the diameter through the origin from the centre of the circle.

Then it will be easily seen that

$$x = a \cos \omega t + a \cos (\omega t - \theta) \ldots\ldots (1),$$

$$y = a \sin \omega t + a \sin (\omega t - \theta) \ldots\ldots (2).$$

From (1) we have

$$dx = a d\theta \sin (\omega t - \theta),$$

and from (2),

$$dy = - a d\theta \cos (\omega t - \theta).$$

Hence from (III.), observing that $X = 0$ and $Y = 0$,

$$\sin (\omega t - \theta) \frac{\delta^2 x}{\delta t^2} - \cos (\omega t - \theta) \frac{\delta^2 y}{\delta t^2} = 0 \ldots\ldots (3).$$

Again, from (1),

$$\frac{\delta x}{\delta t} = - a\omega \sin \omega t + a \left(\frac{\delta\theta}{\delta t} - \omega\right) \sin (\omega t - \theta),$$

$$\frac{\delta^2 x}{\delta t^2} = - a\omega^2 \cos \omega t - a \left(\frac{\delta\theta}{\delta t} - \omega\right)^2 \cos (\omega t - \theta) + a \frac{\delta^2\theta}{\delta t^2} \sin (\omega t - \theta);$$

and from (2),

$$\frac{\delta y}{\delta t} = a\omega \cos \omega t - a \left(\frac{\delta\theta}{\delta t} - \omega\right) \cos (\omega t - \theta),$$

$$\frac{\delta^2 y}{\delta t^2} = - a\omega^2 \sin \omega t - a \left(\frac{\delta\theta}{\delta t} - \omega\right)^2 \sin (\omega t - \theta) - a \frac{\delta^2\theta}{\delta t^2} \cos (\omega t - \theta);$$

and therefore by (3),

$$a\omega^2 \{\sin \omega t \cos (\omega t - \theta) - \cos \omega t \sin (\omega t - \theta)\} + a \frac{\delta^2\theta}{\delta t^2} = 0,$$

$$\omega^2 \sin \theta + \frac{\delta^2\theta}{\delta t^2} = 0:$$

multiplying by $2 \frac{\delta\theta}{\delta t}$, and integrating,

$$\frac{\delta\theta^2}{\delta t^2} = C + 2\omega^2 \cos \theta.$$

But the absolute velocity of the particle being initially zero, it is clear that 2ω will be the initial value of $\frac{\delta\theta}{\delta t}$; and therefore, θ being initially zero, we have

$$4\omega^2 = C + 2\omega^2, \quad C = 2\omega^2,$$

and therefore

$$\frac{\delta\theta^2}{\delta t^2} = 2\omega^2 (1 + \cos \theta) = 4\omega^2 \cos^2 \frac{\theta}{2}, \quad \frac{\delta\theta}{\delta t} = 2\omega \cos \frac{\theta}{2},$$

$$\frac{\cos \frac{\theta}{2}\, \delta\theta}{\cos^2 \frac{\theta}{2}} = 2\omega\delta t, \quad \frac{\delta \sin \frac{\theta}{2}}{1 - \sin^2 \frac{\theta}{2}} = \omega\delta t.$$

Integrating, we have

$$\log \frac{1 + \sin \frac{\theta}{2}}{1 - \sin \frac{\theta}{2}} = 2\omega t + C;$$

but $\theta = 0$ when $t = 0$; hence $C = 0$, and we have

$$\frac{1 + \sin \frac{\theta}{2}}{1 - \sin \frac{\theta}{2}} = \varepsilon^{2\omega t},$$

and therefore

$$\sin \frac{\theta}{2} = \frac{\varepsilon^{\omega t} - \varepsilon^{-\omega t}}{\varepsilon^{\omega t} + \varepsilon^{-\omega t}},$$

which determines the position of the particle within the tube at any time. When $t = \infty$, we have $\sin \frac{\theta}{2} = 1$, and therefore $\theta = \pi$, which shews that, after the lapse of an infinite time, the particle will arrive at the point of rotation.

5. If we pursue the same course as in the solution of the problems (1), (2), (4), we may obtain a convenient formula for the following more general problem: a plane curvilinear tube of any invariable form whatever revolves in its own plane about a fixed point with a uniform angular velocity; to determine the motion of a particle acted on by any forces within the tube.

Let ω be the constant angular velocity of the tube about the fixed point; r the distance of the particle at any time from this point; ϕ the angle between the simultaneous directions of r and of a line joining an assigned point of the tube with the fixed point of rotation; ds an element of the length of the tube at the place of the particle, and S the accelerating force on the particle resolved along the element ds; then the equation for the motion of the particle will be

$$r^2 \frac{\delta\phi^2}{\delta t^2} + \frac{\delta r^2}{\delta t^2} - \omega^2 r^2 = 2\int S \frac{ds}{d\phi}\, \delta\phi\,;$$

but since, the form of the tube being invariable, $\delta\phi$, δr may evidently be replaced by $d\phi$, dr, we have, putting, for the sake of uniformity of notation, dt in place of δt,

$$r^2 \frac{d\phi^2}{dt^2} + \frac{dr^2}{dt^2} - \omega^2 r^2 = 2\int S ds.$$

If ω be zero, the formula will become

$$r^2 \frac{d\phi^2}{dt^2} + \frac{dr^2}{dt^2} = 2\int S\, ds,$$

the well-known formula for the motion of a particle under the action of any forces within an immoveable plane tube.

6. In the foregoing examples the position of the tube varies with the time, the form however remains invariable. We will now give an example in which the form changes with the time.

A particle is projected with a given velocity within a circular tube, the radius of which increases in proportion to the time while the centre remains stationary; to determine the motion of the particle, the tube being supposed to lie always in a horizontal plane.

The equation to the circle will be

$$x^2 + y^2 = a^2 (1 + \alpha t)^2 \ldots\ldots\ldots\ldots\ldots (1),$$

where a and α are some constant quantities; hence

$$x\, dx + y\, dy = 0,$$

and therefore, by the general formula (III),

$$y\frac{\delta^2 x}{\delta t^2} - x\frac{\delta^2 y}{\delta t^2} = 0;$$

integrating, we have

$$y\frac{\delta x}{\delta t} - x\frac{\delta y}{\delta t} = C.$$

Let the axis of x be so chosen as to coincide with the initial distance of the particle from the centre, and let β be the initial velocity of the particle along the tube; then $C = -a\beta$, and therefore

$$x\frac{\delta y}{\delta t} - y\frac{\delta x}{\delta t} = a\beta \;\ldots\ldots\ldots\; (2);$$

again, from (1) we have

$$x\frac{\delta x}{\delta t} + y\frac{\delta y}{\delta t} = a^2\alpha\,(1+\alpha t) \ldots (3);$$

multiplying (2) by y and (3) by x, and subtracting the former result from the latter, we have

$$(x^2+y^2)\frac{\delta x}{\delta t} = a^2\alpha\,(1+\alpha t)\,x - a\beta y,$$

and therefore by (1)

$$a(1+\alpha t)^2\frac{\delta x}{\delta t} = a\alpha\,(1+\alpha t)\,x - \beta\,\{a^2(1+\alpha t)^2 - x^2\}^{\frac{1}{2}}.$$

Put $1+\alpha t = \tau$, then

$$a\alpha\tau^2\frac{\delta x}{\delta\tau} = a\alpha\tau x - \beta\,(a^2\tau^2 - x^2)^{\frac{1}{2}};$$

again, put $x = m\tau$, and there is

$$a\alpha\tau^2\left(m + \tau\frac{\delta m}{\delta\tau}\right) = a\alpha m\tau^2 - \beta\tau\,(a^2-m^2)^{\frac{1}{2}},$$

$$a\alpha\tau^3\frac{\delta m}{\delta\tau} = -\beta\tau\,(a^2-m^2)^{\frac{1}{2}},$$

$$-a\alpha\frac{\delta m}{(a^2-m^2)^{\frac{1}{2}}} = \beta\frac{\delta\tau}{\tau^2};$$

integrating,

$$C + a\alpha\cos^{-1}\frac{m}{a} = -\frac{\beta}{\tau},$$

or, putting for m its value,

$$C + a\alpha \cos^{-1} \frac{x}{a\tau} = -\frac{\beta}{\tau},$$

and putting for τ its value $1 + \alpha t$,

$$C + a\alpha \cos^{-1} \frac{x}{a(1+\alpha t)} = -\frac{\beta}{1+\alpha t}.$$

Now $x = a$ when $t = 0$; hence $C = -\beta$, and therefore

$$a\alpha \cos^{-1} \frac{x}{a(1+\alpha t)} = \frac{\alpha\beta t}{1+\alpha t},$$

$$x = a(1+\alpha t) \cos \frac{\beta t}{a(1+\alpha t)},$$

and therefore from (1)

$$y = a(1+\alpha t) \sin \frac{\beta t}{a(1+\alpha t)};$$

which give the absolute position of the particle at any assigned time.

We proceed now to the consideration of the motion of a particle along a surface from which it is unable to detach itself, while the surface itself changes its position or its form, or both, according to any assigned law. To fix the ideas, we suppose the particle to move between two surfaces indefinitely close together, so as to be expressed by the same equation.

Let x, y, z be the co-ordinates of the particle at any time t; and let $\delta x, \delta y, \delta z$ be the increments of x, y, z, in an indefinitely small time δt; also let dx, dy, dz denote the increments of x, y, z, in passing from the point x, y, z, to any point near to it within the surface as it exists at the time t. Also let X, Y, Z denote the resolved parts of the accelerating forces on the particle at the time t parallel to the axes of x, y, z; then, observing that the action of the surface on the particle is always in the direction of the normal at each point, we have, by D'Alembert's Principle combined with the Principle of Virtual Velocities,

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right) dx + \left(\frac{\delta^2 y}{\delta t^2} - Y\right) dy + \left(\frac{\delta^2 z}{\delta t^2} - Z\right) dz = 0 \ldots\ldots (A).$$

Again, since the position and form of the surface vary according to an assigned law, its equation must evidently be known at any given time, and therefore we must have, from the nature of each particular problem, certain conditions between the quantities x, y, z, t, equivalent to a single equation

$$F = f(x, y, z, t) = 0 \ldots\ldots\ldots\ldots\ldots (B).$$

Taking the total differential of (B), we have

$$\frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dz = 0;$$

eliminating dz between this equation and (A), we get

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right)\frac{dF}{dz}dx + \left(\frac{\delta^2 y}{\delta t^2} - Y\right)\frac{dF}{dz}dy$$
$$= \left(\frac{\delta^2 z}{\delta t^2} - Z\right)\left(\frac{dF}{dx}dx + \frac{dF}{dy}dy\right);$$

but dx and dy are independent quantities, we have therefore, by equating separately their coefficients on each side of the equation,

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right)\frac{dF}{dz} = \left(\frac{\delta^2 z}{\delta t^2} - Z\right)\frac{dF}{dx},$$
$$\left(\frac{\delta^2 y}{\delta t^2} - Y\right)\frac{dF}{dz} = \left(\frac{\delta^2 z}{\delta t^2} - Z\right)\frac{dF}{dy},$$

and therefore also

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right)\frac{dF}{dy} = \left(\frac{\delta^2 y}{\delta t^2} - Y\right)\frac{dF}{dx}:$$

any two of these three relations, together with the equation (B), will give us three equations in x, y, z, t, whence x, y, z are to be determined in terms of t.

The following example will serve to illustrate the use of these equations. We have taken a case where the form of the surface remains invariable, its position alone being liable to change. The analysis however, in the solution of problems of the class which we are considering, receives its general character solely in consequence of the presence of t in the equation (B), and therefore the example which we have chosen is sufficient for the general object we have in view.

A particle descends by the action of gravity down a plane which revolves uniformly about a vertical axis through which it passes; to determine the motion of the particle.

Let the plane of x, y be taken horizontal, the axis of x coinciding with the initial intersection of the revolving plane with the horizontal plane through the origin, and let the axis of z be taken vertically downwards; then, ω denoting the angular velocity of the plane, its equation at any time t will be

$$F = y\cos\omega t - x\sin\omega t = 0 \;\ldots\ldots\ldots\; (1),$$

whence

$$\frac{dF}{dx} = -\sin\omega t,\ \frac{dF}{dy} = \cos\omega t,\ \frac{dF}{dz} = 0;$$

also $X=0$, $Y=0$, $Z=g$; and therefore, from either of the two first of the three general relations,

$$\frac{\delta^2 z}{\delta t^2} = g \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and from the third,

$$\frac{\delta^2 x}{\delta t^2} \cos \omega t + \frac{\delta^2 y}{\delta t^2} \sin \omega t = 0 \ldots\ldots\ldots\ldots\ldots (3).$$

Let r denote the distance of the particle at any time from the axis of z; then

$$x = r \cos \omega t, \quad y = r \sin \omega t,$$

whence

$$\frac{\delta x}{\delta t} = \frac{\delta r}{\delta t} \cos \omega t - \omega r \sin \omega t,$$

$$\frac{\delta^2 x}{\delta t^2} = \frac{\delta^2 r}{\delta t^2} \cos \omega t - 2\omega \frac{\delta r}{\delta t} \sin \omega t - \omega^2 r \cos \omega t,$$

$$\frac{\delta y}{\delta t} = \frac{\delta r}{\delta t} \sin \omega t + \omega r \cos \omega t,$$

$$\frac{\delta^2 y}{\delta t^2} = \frac{\delta^2 r}{\delta t^2} \sin \omega t + 2\omega \frac{\delta r}{\delta t} \cos \omega t - \omega^2 r \sin \omega t;$$

and therefore from (3)

$$\frac{\delta^2 r}{\delta t^2} - \omega^2 r = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Let the initial values of z, $\frac{dz}{dt}$ be 0, β; and those of r, $\frac{\delta r}{\delta t}$ be a, α; then, from the equations (2) and (4), after executing obvious operations, we shall obtain

$$z = \tfrac{1}{2} g t^2 + \beta t,$$

$$2\omega r = (\omega a + \alpha)\,\varepsilon^{\omega t} + (\omega a - \alpha)\,\varepsilon^{-\omega t},$$

and

$$\log \frac{(\omega^2 r^2 + \alpha^2 - \omega^2 a^2)^{\frac{1}{2}} + \omega r}{\alpha + \omega a} = \frac{\omega}{g} \{(2gz + \beta^2)^{\frac{1}{2}} - \beta\};$$

the two first of these equations give the position of the particle on the revolving plane, and therefore, by virtue of the equation (1), the absolute position of the particle at any time; while the third is the equation to the path which the particle describes on the plane.

II.—REMARKS ON THE BINOMIAL THEOREM.*

The proof given by Euler of the Binomial Theorem appears to depend upon a casual result of multiplication, and has sometimes been called *tentative*. This proof may be seen, in an extended form, in M. Lebefure de Fourcy's treatise on Algebra; but the apparent *casualty* still remains. In the following sketch, this defect, for such we may suppose it to be from the remark it has excited, is removed to the extent of placing the general theorem on the same footing as its particular case when the exponent is an integer; and so as to embrace the more extended form just alluded to.

1. In the common multiplication of $a+b$ by itself repeatedly, the process shows that the successive coefficients are as at the side, where each one is made by adding the one vertically above it to the one above it on the left. But if m_n signify the number of ways in which m can be selected out of n, it is obvious that

1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

$$m_n = m_{n-1} + (m-1)_{n-1};$$

whence it easily follows that the m^{th} *variable* number in the n^{th} row is m_n. Hence the binomial theorem easily follows for an integer exponent.

Now let there be a succession of symbols, A, B, C, &c. such that the change which converts A into B, converts B into C, C into D, and so on: and let a, b, c, &c. be another set having the same property. Taking the expression $A+a$, make a succession of similar operations as follows:—each operation consists in making the change with respect to A, and multiplying by A, doing the same with a, and adding the results. Let Θ be the symbol of this operation. We have then

$$\begin{aligned}\Theta(A+a) &= AB + Aa \\ &\qquad + Aa + ab = AB + 2Aa + ab, \\ \Theta^2(A+a) &= ABC + 2ABa + Aab \\ &\qquad + ABa + 2Aab + abc \\ &= ABC + 3ABa + 3Aab + abc,\end{aligned}$$

and so on. When $ABC\ldots.$ and $abc\ldots.$ contain n factors, let them be A_n and a_n; we have then

$$\Theta^n(A+a) = A_n + nA_{n-1}a + n\frac{n-1}{2}A_{n-2}a_2 + \ldots + a_n,$$

which is an extension of the binomial theorem. Observe that

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* From a Correspondent.

the right of $\Theta^n (A+a)$ to the designation $(A+a)_n$ is not universal.

Now let

$$B = A + \delta,\ C = A + 2\delta, \text{ \&c., and } b = a + \delta,\ c = a + 2\delta, \text{ \&c.}$$

we have

$$\Theta(A+a) = (A+a)(A+a+\delta), \text{ \&c. or } \Theta^n(A+a) = (A+a)_n,$$

whence the above theorem is true when P_n means $P^{n|\delta}$, and there readily follows, $1.2.3.\ldots n$ being denoted by $[n]$,

$$\frac{(A+a)_n}{[n]} = \frac{A_n}{[n]} + \frac{A_{n-1}}{[n-1]}\frac{a_1}{[1]} + \frac{A_{n-2}}{[n-2]}\frac{a_2}{[2]} + \ldots + \frac{a_n}{[n]};$$

or the series $1 + (A+a)_1 x + (A+a)_2 \dfrac{x^2}{2} + \ldots$ is the product of the series $1 + A_1 x + A_2 \dfrac{x^2}{2} + \ldots$ and $1 + a_1 x + a_2 \dfrac{x^2}{2} + \ldots$;

from whence, by the usual consideration of the equation $\phi(A+a) = \phi A \times \phi a$, it follows that

$$\left\{1 + Ax + A(A+\delta)\frac{x^2}{2} + \ldots\right\}^m = 1 + mAx + mA(mA+\delta)\frac{x^2}{2} + \ldots$$

for all possible values of m. And $A = 1$ $\delta = -1$ gives the binomial theorem, while $Ax = 1$, $\delta = 0$ gives the exponential theorem.

A. D. M.

III.—ON THE PROPERTIES OF A CERTAIN SYMBOLICAL EXPRESSION.

By Arthur Cayley, B.A. Trinity College.

The series

$$S_{p\,0}^{\infty} \cdot \zeta_p (a^2 + b^2 \ldots n \text{ terms})^{p+i} \left(\frac{l}{1+l} \cdot \frac{d^2}{da^2} + \frac{m}{1+m} \cdot \frac{d^2}{db^2} \cdot\cdot\right)^p \times \frac{1}{\{(1+l)a^2 + (1+m)b^2 \ldots\}^i}$$

$$\left(\zeta_p = \frac{1}{2^{2p+1}\, 1.2\ldots p \,.\, i(i+1)\ldots(i+p)}\right) \ldots\ldots (\psi),$$

possesses some remarkable properties, which it is the object of the present paper to investigate. We shall prove that the symbolical expression (ψ) is independent of a, b, &c., and equivalent to the definite integral

$$\int_0^1 \frac{x^{2i-1}\,dx}{\{(1+lx^2)(1+mx^2)\ldots\}^{\frac{1}{2}}},$$

a property which we shall afterwards apply to the investigation of the attractions of an ellipsoid upon an external point, and to some other analogous integrals. The demonstration of this, which is one of considerable complexity, may be effected as follows:

Writing the symbol $\frac{l}{1+l} \cdot \frac{d^2}{da^2} + \frac{m}{1+m} \cdot \frac{d^2}{db^2} \dots\dots$ under the form

$$\left(\frac{d^2}{da^2} + \frac{d^2}{db^2} + \frac{d^2}{dc^2}\right) - \left(\frac{1}{1+l} \cdot \frac{d^2}{da^2} + \frac{1}{1+m} \cdot \frac{d^2}{db^2} \dots\dots\right)$$

$$= \Delta - \left(\frac{1}{1+l} \cdot \frac{d^2}{da^2} + \frac{1}{1+m} \cdot \frac{d^2}{db^2} \dots\dots\right) \text{ suppose.}$$

Let the p^{th} power of this quantity be expanded in powers of Δ. The general term is

$$(-1)^q \cdot \frac{p \cdot (p-1) \dots (p-q+1)}{1 \cdot 2 \dots q} \cdot \Delta^{p-q} \cdot \left(\frac{1}{1+l} \cdot \frac{d^2}{da^2} \dots\dots\right)^q,$$

which is to be applied to $\frac{1}{\{(1+l)\,a^2 \dots\dots\}^i}$.

Considering the expression

$$\left(\frac{1}{1+l} \cdot \frac{d^2}{da^2} \dots\dots\right)^q \cdot \frac{1}{\{(1+l)\,a^2 \dots\dots\}^i};$$

if for a moment we write $(1+l)\,a^2 = a_1^2$, &c. $\Delta_1 = \frac{d^2}{da_1^2} + \frac{d^2}{db_1^2} \dots$ $\rho_1 = a_1^2 + b_1^2 + c_1^2 \dots\dots$, this becomes

$$\Delta_1^q \cdot \frac{1}{\rho_1^i}.$$

Now it is immediately seen that $\Delta_1 \frac{1}{\rho_1^{i'}} = \frac{2i' \cdot (2i' + 2 - n)}{\rho_1^{i'+1}}$; from which we may deduce

$$\Delta_1^q \cdot \frac{1}{\rho_1^i} = \frac{2i \cdot (2i+2) \dots (2i+2q-2)\,(2i+2-n) \dots (2i+2q-n)}{\rho_1^{i+q}},$$

or, restoring the value of ρ_1, and forming the expression for the general term of (ψ), this is

$$\zeta_p \cdot \rho^{n+1} \left\{ \begin{array}{l} \Delta^p \cdot \dfrac{1}{(a^2 + b^2 \dots + la^2 + mb^2 + \&c.)^i} \\[2ex] -\dfrac{p}{1}\, 2i \cdot (2i + 2 - n)\, \Delta^{p-1} \cdot \dfrac{1}{(a^2 + b^2 + \dots + la^2 + mb^2)^i} \\[2ex] + \&c. \end{array} \right.$$

ρ representing the quantity $a^2 + b^2 + \&c.$

Hence, selecting the terms of the s^{th} order in l, m, &c. the expression for the part of (ψ) which is of the s^{th} order in l, m, &c. may be written under the form

$$S_{p\,0}^{\ s} \frac{(-1)^p \rho^{p+i} \zeta_p}{1 \,.\, 2 \ldots\ldots s}$$

multiplied by

$$\left\{\begin{array}{l} i\,.\,(i+1) \ldots\ldots (i+s-1)\, \Delta^p \,.\, \dfrac{U}{\rho^{i+s}} \\ -\dfrac{p}{1}\, 2i\,.\,(2i+2-n)\,(i+1) \ldots\ldots (i+s)\, \Delta^{p-1}\, \dfrac{U}{\rho^{i+s+1}} \\ +\dfrac{p\,.\,p-1}{1\,.\,2}\, 2i.(2i+2)\,(2i+2-n)\,(2i+4-n)\,(i+2) \ldots (i+s+1)\, \Delta^{p-2} \dfrac{U}{\rho^{i+s+2}} \\ -\ \&\text{c.} \qquad\qquad [la^2 + mb^2 \ldots\ldots = U \text{ suppose}] \end{array}\right.$$

which for conciseness we shall represent by

$$\frac{(-1)^p}{1\,.\,2 \ldots\ldots s} S_{p\,0}^{\ s}\, \rho^{p+i} \,.\, \zeta_p \left\{\begin{array}{l} a_s\, \Delta^p \,.\, \dfrac{U}{\rho^{i+s}} \\ -\dfrac{p}{1} \beta_s\, \Delta^{p-1} \,.\, \dfrac{U}{\rho^{i+s+1}} \\ +\dfrac{p\,.\,p-1}{1\,.\,2}\, \gamma_s \,.\, \Delta^{p-2} \,.\, \dfrac{U}{\rho^{i+s+2}} \\ -\ \&\text{c.} \end{array}\right.$$

$= S$ suppose.

Now U representing any homogeneous function of the order $2s$, it is easily seen that

$$\Delta \frac{U}{\rho^i} = \frac{\Delta U}{\rho^i} + 2i\,.\,(2i+2-4s-n)\,\frac{U}{\rho^{i+1}}.$$

And repeating continually the operation Δ, observing that ΔU, $\Delta^2 U$, &c. are of the orders $2(s-1)$, $2(s-2)$, &c. we at length arrive at

$$\begin{aligned} \Delta^q \,.\, \frac{U}{\rho^i} = {} & \Delta^q \,.\, U \,.\, \frac{1}{\rho^i} \\ & + \frac{q}{1}\, 2i.(2i+2q-4s-n)\, \Delta^{q-1} U . \frac{1}{\rho^{i+1}} \\ & + \frac{q.q-1}{1.2}\, 2i.(2i+2)\,(2i+2q-4s-n)\,(2i+2q-4s-n-2)\, \Delta^{q-2} . U . \frac{1}{\rho^{i+2}} \\ & \quad \vdots \\ & + 2i\,.\,(2i+2) \ldots (2i+2q)\,(2i+2q-4s-n) \ldots (2i+2-4s-n)\, \frac{U}{\rho^{i+q}}. \end{aligned}$$

Changing i into $s+i+i'$, we have an equation which we may represent by

$$\Delta^q \cdot \frac{U}{\rho^{s+i+i'}} = A_{q,i'} \frac{\Delta^q \cdot U}{\rho^{s+i+i'}} + {}^1A_{q,i'} \frac{\Delta^{q-1} \cdot U}{\rho^{s+i+i'+1}} \cdot\cdot + {}^qA_{q,i'} \frac{U}{\rho^{s+i+i'+q}} \cdot\cdot (a),$$

where in general

$${}^rA_{q,i'} = \frac{q \cdot (q-1) \ldots\ldots (q-r+1)}{1.2 \ldots\ldots r}$$

$$\times (2s+2i+2i')(2s+2i'+2) \ldots\ldots (2s+2i'+2i+2r-2)$$

$$\times (2i+2i'+2q-2s-\mathfrak{n}) \ldots\ldots (2i+2i'+2q-2s-\mathfrak{n}-2r+2).$$

Now the value of S, written at full length, is

$$\frac{(-1)^s}{1.2 \ldots\ldots s} \left\{ \begin{array}{l} \zeta_s \rho^{s+i} \left(a_s \Delta^s \dfrac{U}{\rho^{s+i}} + \dfrac{s}{1} \beta_s \Delta^{s-1} \dfrac{U}{\rho^{s+i+1}} \ldots\ldots \right. \\ + \zeta_{s-1} \rho^{s+i-1} \left(a_s \Delta^{s-1} \dfrac{U}{\rho^{s+i}} + \dfrac{s-1}{1} \beta_s \Delta^{s-2} \dfrac{U}{\rho^{s+i+1}} + \ldots\ldots \right. \\ + \text{\&c.} \end{array} \right.$$

and substituting for the several terms of this expansion the values given by the equation (a), we have

$$S = \frac{(-1)^s}{1.2 \ldots\ldots s} \left(k_0 \Delta^s U + k_1 \cdot \frac{1}{\rho} \Delta^{s-1} U \ldots\ldots + k_s \frac{1}{\rho^s} \cdot U \right)$$

where in general

$$k_x = a_s \left({}^xA_{s,0} \zeta_s + {}^{x-1}A_{s-1,0} \zeta_{s-1} \ldots\ldots + A_{s-x,0} \zeta_{s-x} \right)$$

$$+ \beta_s \left(\frac{s}{1} {}^{x-1}A_{s-1,1} \zeta_s \ldots\ldots + A_{s-x,1} \frac{(s-x+1)}{1} \zeta_{s-x+1} \right)$$

$$\vdots$$

$$+ \lambda_s \left(\frac{s \cdot (s-1) \ldots\ldots (s-x+1)}{1.2 \ldots\ldots x} A_{s-x,x} \right),$$

λ_s being the $(x+1)^{\text{th}}$ of the series $a_s, \beta_s \ldots\ldots$

Substituting for the quantities involved in this expression, and putting, for simplicity, $2i+2-\mathfrak{n}=2\gamma$, we have, without any further reduction, except that of arranging the factors of the different terms, and cancelling those which appear in the numerator and denominator of the same term,

$$\frac{(-1)^s k_x}{1.2 \ldots\ldots s} =$$

$$\frac{(-1)^{s-x}(1-\gamma)(2-\gamma) \ldots\ldots (\lambda-\gamma)}{2^{2s+1} \cdot 1.2 \ldots\ldots s \cdot 1.2 \ldots\ldots (s-x) \cdot 1.2 \ldots\ldots x}$$

multiplied by the series

$$(i+s+1)\ldots(i+s+x-1)\left\{1+\frac{\gamma x}{1.(x-\gamma)}+\frac{1}{1.2}\,\frac{\gamma.(\gamma+1)\,x.(x-1)}{(x-\gamma)(x-1-\gamma)}+\ldots(x+1)\text{ terms}\right.$$

$$-\frac{(i+s)\ldots(i+s+x-2)}{1-\gamma}\left\{x+\frac{\gamma x.(x-1)}{1.(x-\gamma)}+\frac{1}{1.2}\,\frac{\gamma.(\gamma+1)\,.\,x.(x-1)(x-2)}{(x-\gamma)(x-1-\gamma)}\right.$$

$$+\ldots x\text{ terms}$$

$$+(-1)^r\frac{(i+s-r+1)\ldots(i+s+x-r-1)}{(1-\gamma)(2-\gamma)\ldots(r-\gamma)}\left\{x.(x-1)\ldots(x-r+1)+\frac{\gamma}{1}\,\frac{x.(x-1)\ldots(x-r)}{(x-\gamma)}\right.$$

$$\ldots x+1-r\text{ terms}$$

to $r=x$.

Now it may be shown that

$$\frac{1}{(1-\gamma)(2-\gamma)\ldots\ldots(r-\gamma)}$$

$$\left\{x.(x-1)\,.\,.(x-r+1)+\frac{\gamma}{1}\,\frac{x.(x-1)\,.\,.\,.(x-r)}{x-\gamma}+\&c.\,.\,.(x+1-r)\text{ terms}\right\}$$

$$=\frac{x.(x-1)\ldots\ldots(r+1)\,.\,x.(x-1)\ldots\ldots(x-r+1)}{(1-\gamma)(2-\gamma)\ldots\ldots(x-\gamma)},$$

which reduces the expression for k_x to the form

$$\frac{(-1)^s.k_x}{1.2.\,.s}=\frac{(-1)^{s+x}}{2^{2s+1}.1.2.\,.s.1.2.\,.(s-x)}\left\{\begin{array}{l}(i+s+1).\,.(i+s+x-1)\\ -(i+s).\,.(i+s+x-2)\dfrac{x}{1}\\ +(i+s-1).\,.(i+s+x-3)\dfrac{x.(x-1)}{1.2}\\ \quad\&c.\ (x+1)\text{ terms};\end{array}\right.$$

from which it may be shown, that except for $x=0$, $k_x=0$.

The value $x=0$, observing that the expression

$$(i+s+1)(i+s+2)\ldots\ldots(i+s-1)$$

represents $\frac{1}{i+s}$, gives

$$\frac{(-1)^s k_0}{1.2\ldots\ldots s}=\frac{(-1)^s}{2^{2s}(1.2\ldots\ldots s)^2.(2i+2s)};$$

or we have simply

$$S=\frac{(-1)^s}{2^{2s}(1.2\ldots\ldots s)^2.(2i+2s)}\Delta^s.U,$$

where $\Delta = \frac{d^2}{da^2} + \frac{d^2}{db^2} + \ldots,\quad U = (la^2 + mb^2 \ldots)^s$.

Consider the term $\frac{1.2 \ldots s}{1.2 \ldots \lambda \,.\, 1.2 \ldots \mu \,.\, \&c.}\, a^{2\lambda} \,.\, b^{2\mu} \ldots l^\lambda \,.\, m^\mu \ldots$

With respect to this, Δ^s reduces itself to

$$\frac{1.2 \ldots s}{1.2 \ldots \lambda \,.\, 1.2 \ldots \mu \,.\, \&c.} \left(\frac{d}{da}\right)^{2\lambda} \ldots$$

and the corresponding term of S is

$$\frac{(-1)^s}{2^{2s}.(2i+2s)(1.2 \ldots \lambda(1.2 \ldots \mu \,\&c.)^2}\, 1.2 \ldots 2\lambda \,.\, 1.2 \ldots 2\mu \,\&c.\; l^\lambda . m^\mu \ldots$$

$$= \frac{(-1)^s \,.\, 1.3 \ldots (2\lambda - 1) \,.\, 1.3 \ldots (2\mu - 1)\,\&c.}{(2i + 2s)\, 2.4 \ldots 2\lambda \,.\, 2.4 \ldots 2\mu \,\&c.}\, l^\lambda . m^\mu \ldots$$

which, omitting the factor $\frac{1}{2i + 2s}$, and multiplying by x^{2s}, is the general term of the s^{th} order in l, m, n, of

$$\frac{1}{\sqrt{\{(1 + lx^2)(1 + mx^2) \ldots\}}}.$$

The term itself is therefore the general term of

$$\int_0^1 \frac{x^{2i-1}\,dx}{\sqrt{\{(1 + lx^2)(1 + mx^2) \ldots}};$$

or taking the sum of all such terms for the complete value of S, and the sum of the different values of S for s variable, we have the required equation

$$\psi = \int \frac{x^{2i-1} \,.\, dx}{\sqrt{\{(1 + lx^2)(1 + mx^2) \ldots}}.$$

Another and perhaps more remarkable form of this equation may be deduced by writing $\frac{a^2}{1+l}$, $\frac{b^2}{1+m}$, &c. for a^2, b^2, &c., and putting $\frac{a^2}{1+l} + \frac{b^2}{1+m} + \&c. = \eta^2$, $l\eta^2 = \alpha^2$, $m\eta^2 = \beta^2$, &c., we readily deduce

$$\eta^{n-2i} . \int_0^1 \frac{x^{2i-1} \,.\, dx}{\{(\eta^2 + \alpha^2 x^2)(\eta^2 + \beta^2 x^2) \ldots\}^{\frac{1}{2}}}$$

$$= S_p{}_0^\infty \frac{1}{2^{2p+1} \,.\, 1.2 \ldots p . i . (i+1) \ldots (i+p)} \left(\alpha^2 \frac{d^2}{da^2} + \beta^2 . \frac{d^2}{db^2} \ldots\right)^p \frac{1}{(a^2+b^2+c^2)^i},$$

η being determined by the equation

$$\frac{a^2}{\eta^2 + \alpha^2} + \frac{b^2}{\eta^2 + \beta^2} \ldots = 1;$$

or, as it may otherwise be written,

$$\eta^2 = a^2 + b^2 + c^2 - \frac{a^2\alpha^2}{\eta^2 + \alpha^2} - \frac{b^2\beta^2}{\eta^2 + \beta^2} - \&c.$$

n, it will be recollected, denotes the number of the quantities a, b, &c.

Now suppose

$$V = \iint \ldots\ldots \phi(a - x,\ b - y \ldots\ldots)\, dx\, dy \ldots\ldots$$

(the integral sign being repeated n times) where the limits of the integral are given by the equation

$$\frac{x^2}{h^2} + \frac{y^2}{h_{,}^2} + \&c. = 1;$$

and that it is permitted, throughout the integral to expand the function $\phi(a - x, \ldots\ldots)$ in ascending powers of x, y, &c. (the condition for which is apparently that of ϕ not becoming infinite for any values of x, y, &c., included within the limits of the integration): then observing that any integral of the form $\iint \ldots\ldots x^p y^q \ldots dx\, dy$ &c.. where either p, q, &c... is odd, when taken between the required limits contains equal positive and negative elements, and therefore vanishes, the general term of V assumes the form

$$\frac{1}{1.2..2r.1.2..2s..}\left(\frac{d}{da}\right)^{2r}\left(\frac{d}{db}\right)^{2s}..\phi(a,b..)\iint..x^{2r}y^{2s}..dx\,dy..$$

Also, by a formula quoted in the eleventh number of the *Mathematical Journal*, the value of the definite integral $\iint..x^{2r}y^{2s}..dx\,dy..$is

$$h^{2r+1}.h_{,}^{2s+1}\ldots\ldots\frac{\Gamma(r+\frac{1}{2}).\Gamma(s+\frac{1}{2})\ldots\ldots}{\Gamma(r+s+\ldots\ldots+\frac{n}{2}+1)},$$

(observing that the value there given referring to positive values only of the variables, must be multiplied by 2^n): or, as it may be written,

$$h^{2r+1}.h_{,}^{2s+1}\ldots\ldots\pi^{\frac{1}{2}n}.\frac{1}{2^{r+s}}\ldots\ldots\frac{1.3.\ldots(2r-1)\,1.3.\ldots(2s-1)..}{\frac{n}{2}.\left(\frac{n}{2}+1\right)\ldots\left(\frac{n}{2}+r+s\right)\ldots\Gamma\left(\frac{n}{2}\right)},$$

whence the general term of V takes the form

$$\frac{hh_{,}..\pi^{\frac{1}{2}n}}{\Gamma\left(\frac{n}{2}\right)}.\frac{1}{\frac{n}{2}.\left(\frac{n}{2}+1\right)..\left(\frac{n}{2}+r+s\right)..}.\frac{1}{2^{2r+2s..}}\,\frac{1}{1.2.3..r.1.2..s..}$$

$$\times\left(h^2.\frac{d^2}{da^2}\right)^r.\left(h_{,}^2\frac{d^2}{db^2}\right)^s\ldots\ldots\phi(a,b,\ldots\ldots).$$

And putting $r + s + \&c. = p$, and taking the sum of the terms that answer to the same value of p, it is immediately seen that this sum is

$$= \frac{hh_, \ldots \pi^{\frac{1}{2}n}}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{1}{2^{2p} \cdot 1.2 \ldots p \cdot \frac{n}{2} \cdot \left(\frac{n}{2}+1\right) \ldots \left(\frac{n}{2}+p\right)}$$

$$\left(h^2 \frac{d^2}{da^2} + h_,^2 \cdot \frac{d^2}{db^2} \cdots\right)^p \cdot \phi(a, b \ldots).$$

Or the function $\phi(a - x,\ b - y \ldots)$ not becoming infinite within the limits of the integration, we have

$$\iint \ldots \phi(a - x,\ b - y \ldots)\, dx\, dy \ldots$$

$$= \frac{2hh_, \ldots \pi^{\frac{1}{2}n}}{\Gamma\left(\frac{n}{2}\right)} S_{p\,0}^{\,\infty} \frac{1}{2^{2p+1} \cdot 1.2 \ldots p \cdot \frac{n}{2} \cdot \left(\frac{n}{2}+1\right) \cdot \left(\frac{n}{2}+p\right)} \left(h^2 \frac{d^2}{da^2} \ldots\right)^p \phi(a, b \ldots).$$

The integral on the first side of the equation extending to all real values of x, y, &c., subject to $\frac{x^2}{h^2} + \frac{y^2}{h_,^2} + \ldots < 1$. Suppose in the first place $\phi(a, b \ldots) = \frac{1}{(a^2 + b^2 \ldots)^{\frac{1}{2}n}}$.

By a preceding formula the second side of the equation reduces itself to

$$\frac{2hh_, \ldots \pi^{\frac{1}{2}n}}{\Gamma\left(\frac{n}{2}\right)} \cdot \int_0^1 \frac{x^{n-1} \cdot dx}{\sqrt{\{(\eta^2 + h^2x^2)(\eta^2 + h_,^2x^2) \ldots (n \text{ factors})\}}},$$

η being given by $\frac{a^2}{\eta^2 + h^2} + \frac{b^2}{\eta^2 + h_,^2} \ldots = 1$.

Hence the formula

$$\iint \ldots n \text{ times } \frac{dx\, dy \ldots}{\{(a-x)^2 + (b-y)^2 \ldots\}^{\frac{1}{2}n}}$$

$$= \frac{2hh_, \ldots \pi^{\frac{1}{2}n}}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 \frac{x^{n-1} \cdot dx}{\sqrt{\{(\eta^2 + h^2x^2)(\eta^2 + h_,^2x^2) \ldots (n \text{ factors})\}}}.$$

The integral on the first side of the equation extending to all real values of x, y, &c. satisfying $\frac{x^2}{h^2} + \frac{y^2}{h_,^2} + \&c. \ldots < 1$; η^2 we have seen being determined by

$$\frac{a^2}{\eta^2 + h^2} + \frac{b^2}{\eta^2 + h_,^2} + \&c. = 1.$$

Finally, the condition of $\phi(a-x,\ b-y.\ldots)$ not becoming infinite within the limits of the integration, reduces itself to $\frac{a^2}{h^2}+\frac{b^2}{h_{,}^2}+\ldots.>1$, which must be satisfied by these quantities.

Suppose in the next place the function $\phi(a, b.\ldots)$ satisfies $\frac{d^2\phi}{da^2}+\frac{d^2\phi}{db^2}+\&c.=0$. The factor $(h^2\frac{d^2}{da^2}+\&c.)$ may be written under the form

$$(h_{,}^2-h^2)\frac{d^2}{db^2}+(h_{,,}^2-h^2)\frac{d^2}{dc^2}+\&c.+h^2\left(\frac{d^2}{da^2}+\frac{d^2}{db^2}\ldots.\right)$$

$$=(h_{,}^2-h^2)\frac{d^2}{db^2}+\&c.$$

Since, as applied to the function ϕ, $\frac{d^2}{da^2}+\frac{d^2}{db^2}+\&c.$ is equivalent to 0, we have in this case

$$\iint\ldots.\ \phi(a-x,\ b-y\ \ldots.)\,dx\,dy\ \ldots.$$

$$=\frac{2hh_{,}\ldots.\ \pi^{\frac{1}{2}n}}{\Gamma\left(\frac{n}{2}\right)}\cdot S_{p\,0}^{\ \infty}\frac{1}{2^{2p+1}\,.\,1.2\ \ldots.\ p\,\frac{n}{2}\,..\left(\frac{n}{2}+p\right)}$$

$$\left\{(h_{,}^2-h^2)\frac{d^2}{db^2}+\ldots.\right\}^p.\ \phi(a, b\ \ldots.);$$

or the first side divided by $hh_{,}\ldots.$ has the remarkable property of depending on the differences $h_{,}^2-h^2$, &c. only the generalisation of a well known property of the function V, in the theory of the attraction of a spheroid upon an external point. If in this equation we put $\phi(a, b.\ldots)=\frac{a}{(a^2+b^2\ \ldots)^n}$, which satisfies the required condition $\frac{d^2\phi}{da^2}+\&c.=0$, we have transferring (a) to the left hand side of the sign S, and putting in a preceding formula, $\alpha^2=0$, $\beta^2=h_{,}^2-h^2$, &c. and η^2+h^2 for η^2,

$$\iint\ldots.\ (n\text{ times})\,.\,\frac{(a-x)\,dx\,dy.\ldots.}{\{(a-x)^2+(b-y)^2\ \ldots.\}^{\frac{1}{2}n}}$$

$$=\frac{2hh_{,}\ldots.\pi^{\frac{1}{2}n}\,a}{\sqrt{(\eta^2+h^2)}\,.\,\Gamma\left(\frac{n}{2}\right)}$$

$$\int_0^1\frac{x^{n-1}\,.\,dx}{[\{\eta^2+h^2+(h_{,}^2-h^2)\,x^2\}.\{\eta^2+h^2+(h_{,,}^2-h^2)\,x^2\}\ \ldots(n-1)\text{ factors}]^{\frac{1}{2}}},$$

where, as before, the integrations on the first side extend to all real values of x, y, &c., satisfying $\frac{x^2}{h^2} + \frac{y^2}{h_{\prime}^2} \dots < 1$; η^2 is determined by $\frac{a^2}{\eta^2 + h^2} + \text{\&c.} = 1$. And $a, b \dots h, h_1$ &c. are subject to $\frac{a^2}{h^2} + \frac{b^2}{h_{\prime}^2} + \text{\&c.} \dots > 1$.

For $n = 3$, this becomes,

$$\iiint \frac{(a-x)\,dx\,dy\,dz}{\{(a-x)^2 + (b-y)^2 + (c-z)^2\}^{\frac{3}{2}}} =$$

$$\frac{4\pi h h_{\prime} h_{\prime\prime}}{\sqrt{(h^2+\eta^2)}} \cdot \int_0^1 \frac{x^2 dx}{\sqrt{[\{\eta^2 + h^2 + (h_{\prime}^2 - h^2)x^2\}\{\eta^2 + h^2 + (h_{\prime\prime}^2 - h^2)x^2\}]}}:$$

the integrations on the first side extending over the ellipsoid whose semiaxes are $h, h_{\prime}, h_{\prime\prime}$, and the point whose coordinates are a, b, c, being exterior to this ellipsoid. Also $\frac{a^2}{\eta^2 + h^2} + \frac{b^2}{\eta^2 + h_{\prime}^2} + \frac{c^2}{\eta^2 + h_{\prime\prime}^2} = 1$: a known theorem.

IV.—ON THE UNIFORM MOTION OF HEAT IN HOMOGENEOUS SOLID BODIES, AND ITS CONNECTION WITH THE MATHEMATICAL THEORY OF ELECTRICITY.*

[Since the following article was written, the writer finds that most of his ideas have been anticipated by M. Chasles in two Mémoires in the *Journal de Mathématiques;* the first in Vol. III., on the Determination of the Value of a certain Definite Integral, and the second, in Vol. V., on a new Method of Determining the attraction of an Ellipsoid on a Point without it. In the latter of these Mémoires, M. Chasles refers to a paper, by himself, in the twenty-fifth *Cahier* of the *Journal de l'Ecole Polytechnique,* in which it is probable there are still farther anticipations, though the writer of the present article has not had access to so late a volume of the latter journal. Since, however, most of his methods are very different from those of M. Chasles, which are nearly entirely geometrical, the following article may be not uninteresting to some readers.]

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* From a Correspondent.

If an infinite homogeneous solid be submitted to the action of certain constant sources of heat, the stationary temperature at any point will vary according to its position; and, through every point there will be a surface, over the whole extent of which, the temperature is constant, which is therefore called an *isothermal* surface. In this paper the case will be considered, in which these surfaces are finite, and consequently closed.

It is obvious that the temperature of any point without a given isothermal surface, depends merely on the form and temperature of the surface, being independent of the actual sources of heat by which this temperature is produced, provided there are no sources without the surface. The temperature of an external point is consequently the same as if all the sources were distributed over this surface, in such a manner as to produce the given constant temperature. Hence we may consider the temperature of any point without the isothermal surface, as the sum of the temperatures due to certain constant sources of heat, distributed over that surface.

To find the temperature produced by a single source of heat, let r be the distance of any point from it, and let v be the temperature at that point. Then, since the temperature is the same for all points situated at the same distance from the source, it is readily shown, that v is determined by the equation

$$-r^2 \frac{dv}{dr} = A.$$

Dividing both members by r^2, and integrating, we have

$$v = \frac{A}{r} + C.$$

Now, let us suppose, that the natural temperature of the solid, or the temperature at an infinite distance from the source, is zero: then we shall have $C = 0$, and consequently

$$v = \frac{A}{r} \quad \ldots\ldots \; (1).$$

Hence, that part of the temperature of a point without an isothermal surface which is due to the sources of heat situated on any element, $d\omega_1^2$, of the surface, is $\frac{\rho_1 d\omega_1^2}{r_1}$, where r_1 is the distance from the element to that point, and ρ_1 a quantity measuring the intensity of the sources of heat, at different parts of the surface. Hence, the supposition being still made,

that there are no sources of heat without the surface, if v be the temperature at the external point, we have

$$v = \iint \frac{\rho_1 d\omega_1^2}{r_1} \ \ldots\ldots \ (2),$$

the integrals being extended over the whole surface. The quantity ρ_1 must be determined by the condition

$$v = v_1 \ \ldots\ldots (3),$$

for any point in the surface, v_1 being a given constant temperature.

Let us now consider what will be the temperature of a point within the surface, supposing all the sources of heat by which the surface is retained at the temperature v_1, to be distributed over it. Since there are no sources in the interior of the surface, it follows, that as much heat must flow out from the interior across the surface, as flows into the interior, from the sources of heat at the surface. Hence the total flux of heat from the original surface, to an adjacent isothermal surface, in the interior, is nothing. Hence also the flux of heat from this latter surface, to an adjacent isothermal surface, in its interior, must be nothing; and so on through the whole of the body within the original surface. Hence the temperature in the interior is constant, and equal to v_1, and therefore, for points at the surface, or within it, we have

$$\iint \frac{\rho_1 d\omega_1^2}{r_1} = v_1 \ \ldots\ldots \ (4).$$

Now, if we suppose the surface to be covered with an attractive medium, whose density at different points is proportional to ρ_1, $-\dfrac{d}{dx}\iint \dfrac{\rho_1 d\omega_1^2}{r_1}$ will be the attraction, in the direction of the axis of x, on a point whose rectangular co-ordinates are x, y, z. Hence it follows, that the attraction of this medium on a point within the surface is nothing, and consequently ρ_1 is proportional to the intensity of electricity, in a state of equilibrium on the surface, the attraction of electricity in a state of equilibrium being nothing on an interior point. Since, at the surface, the value of $\iint \dfrac{\rho_1 d\omega_1^2}{r_1}$ is constant, and since, on that account, its value within the surface is constant also, it follows, that if the attractive force on a point at the surface is perpendicular to the surface, the attraction on a point within the surface is nothing. Hence the sole condition of

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equilibrium of electricity, distributed over the surface of a body, is, that it must be so distributed that the attraction on a point at the surface, oppositely electrified, may be perpendicular to the surface.

Since, at any of the isothermal surfaces, v is constant, it follows, that $-\frac{dv}{dn}$, where n is the length of a curve which cuts all the surfaces perpendicularly, measured from a fixed point to the point attracted, is the total attraction on the latter point; and that this attraction is in a tangent to the curve n, or in a normal to the isothermal surface passing through the point. For the same reason also, if ρ_1 represent a flux of heat, and not an electrical intensity, $-\frac{dv}{dn}$ will be the total flux of heat at the variable extremity of n, and the direction of this flux will be along n, or perpendicular to the isothermal surface. Hence, if a surface in an infinite solid be retained at a constant temperature, and if a conducting body, bounded by a similar surface, be electrified, the flux of heat, at any point, in the first case, will be proportional to the attraction on an electrical point, similarly situated, in the second; and the direction of the flux will correspond to that of the attraction.

Let $-\frac{dv_1}{dn_1}$ be the external value of $-\frac{dv}{dn}$, at the original surface, or the attraction on a point without it, and indefinitely near it. Now this attraction is composed of two parts; one the attraction of the adjacent element of the surface; and the other the attraction of all the rest of the surface. Hence, calling the former of these a, and the latter b, we have

$$-\frac{dv_1}{dn_1} = a + b.$$

Now, since the adjacent element of the surface may be taken as infinitely larger, in its linear dimensions, than the distance from it of the point attracted, its attraction will be the same as that of an infinite plane, of the density ρ_1. Hence a is independent of the distance of the point from the surface, and is equal to $2\pi\rho_1$. Hence

$$-\frac{dv_1}{dn_1} = 2\pi\rho_1 + b.$$

Now, for a point within the surface, the attraction of the adjacent element will be the same, but in a contrary direction, and the attraction of the rest of the surface will be the same, and in the same direction. Hence the attraction on a point within

the surface, and indefinitely near it, is $-2\pi\rho_1 + b$; and consequently, since this is equal to nothing, we must have $b = 2\pi\rho_1$, and therefore

$$-\frac{dv_1}{dn_1} = 4\pi\rho_1 \dots . (5).$$

Hence ρ_1 is equal to the total flux of heat, at any point of the surface, divided by 4π.

It also follows, that if the attraction of matter spread over the surface be nothing on an interior point, the attraction on an exterior point, indefinitely near the surface, is perpendicular to the surface, and equal to the density of the matter at the part of the surface adjacent to that point, multiplied by 4π.

If v be the temperature at any isothermal surface, and ρ the intensity of the sources at any point of this surface, which would be necessary to sustain the temperature v, we have, by (5),

$$-\frac{dv}{dn} = 4\pi\rho,$$

which equation holds, whatever be the manner in which the actual sources of heat are arranged, whether over an isothermal surface, or not; and the temperature produced, in an external point, by the former sources, is the same as that produced by the latter. Also, the total flux of heat across the isothermal surface, whose temperature is v, is equal to the total flux of heat from the actual sources. From this, and from what has been proved above, it follows, that if a surface be described round a conducting or non-conducting electrified body, so that the attraction on points situated on this surface may be every where perpendicular to it, and if the electricity be removed from the original body, and distributed in equilibrium over this surface, its intensity at any point will be equal to the attraction of the original body on that point, divided by 4π, and its attraction on any point without it will be equal to the attraction of the original body on the same point.

If we call E the total expenditure of heat, or the whole flux across any isothermal surface, we have, obviously,

$$E = -\iint \frac{dv_1}{dn_1}\, d\omega^2{}_1.$$

Now this quantity should be equal to the sum of the expenditures of heat from all the sources. To verify this, we must, in the first place, find the expenditure of a single source. Now the temperature produced by a single source is, by (1),

$v = \frac{A}{r}$, and hence the expenditure is obviously equal to $-\frac{dv}{dr} \times 4\pi r^2$, or to $4\pi A$. If $A = \rho_1\, d\omega_1^{\,2}$, this becomes $4\pi\rho_1 d\omega_1^{\,2}$. Hence the total expenditure is $\int\int 4\pi\rho_1\, d\omega_1^{\,2}$, or $-\int\int \frac{dv_1}{dn_1}\, d\omega_1^{\,2}$, which agrees with the expression found above.

The following is an example of the application of these principles.

Uniform Motion of Heat in an Ellipsoid.

The principles established above, afford an easy method of determining the isothermal surfaces, and the corresponding temperatures, in the case in which the original isothermal surface is an ellipsoid.

The first step is to find ρ_1, which is proportional to the quantity of matter at any point in the surface of an ellipsoid, when the matter is so distributed, that the attraction on a point within the ellipsoid is nothing. Now the attraction of a shell, bounded by two concentric similar ellipsoids, on a point within it, is nothing. If the shell be infinitely thin, its attraction will be the same as that of matter distributed over the surface of one of the ellipsoids, in such a manner, that the quantity at any point is proportional to the thickness of the shell at the same point. Let a_1, b_1, c_1, be the semiaxes of one of the ellipsoids, $a_1 + \delta a_1$, $b_1 + \delta b_1$, $c_1 + \delta c_1$, those of the other. Let also p_1 be the perpendicular from the centre to the tangent plane, at any point on the first ellipsoid, and $p_1 + \delta p_1$ the perpendicular from the centre to the tangent plane, at a point similarly situated on the second. Then δp_1 is the thickness of the shell, since, the two ellipsoids being similar, the tangent planes at the points similarly situated on their surfaces, are parallel. Also, on account of their similarity, $\frac{\delta a_1}{a_1} = \frac{\delta b_1}{b_1} = \frac{\delta c_1}{c_1} = \frac{\delta p_1}{p_1}$, and consequently the thickness of the shell is proportional to p_1. Hence we have, by (5),

$$-\frac{1}{4\pi}\frac{dv_1}{dn_1} = \rho_1 = k_1 p_1 \;\ldots\ldots\; (a),$$

where k_1 is a constant, to be determined by the condition $v = v_1$, at the surface of the ellipsoid.

To find the equation of the isothermal surface at which the temperature is $v_1 + dv_1$, let $-dv_1 = C$, in (a). Then we have

$k_1\, p_1\, dn_1 = \frac{C}{4\pi}$, or $p_1\, dn_1 = \theta_1$, where θ_1 is an infinitely small constant quantity; and the required equation will be the equation of the surface traced by the extremity of the line dn_1, drawn externally perpendicular to the ellipsoid. Let x', y', z', be the co-ordinates of any point in that surface, and x, y, z, those of the corresponding points in the ellipsoid. Then, calling α_1, β_1, γ_1, the angles which a normal to the ellipsoid at the point whose co-ordinates are x, y, z, makes with these co-ordinates, and supposing the axes of x, y, z, to coincide with the axes of the ellipsoid, $2a_1$, $2b_1$, $2c_1$, respectively, we have

$$x' - x = dn_1 \cos \alpha_1 = \frac{\frac{x}{a_1^2}\, dn_1}{\sqrt{\left(\frac{x^2}{a_1^4} + \frac{y^2}{b_1^4} + \frac{z^2}{c_1^4}\right)}} = \frac{x}{a_1^2} p_1\, dn_1 = \frac{x}{a_1^2}\, \theta_1.$$

or $x' - x = \frac{x'}{a_1^2}\, \theta_1$, since θ_1 is infinitely small, and therefore also $x' - x$; whence

$$x = x'\left(1 - \frac{\theta_1}{a_1^2}\right) = \frac{x'}{1 + \frac{\theta_1}{a_1^2}}.$$

In a similar manner we should find

$$y = \frac{y'}{1 + \frac{\theta_1}{b_1^2}}, \qquad z = \frac{z'}{1 + \frac{\theta_1}{c_1^2}}.$$

But $\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} + \frac{z^2}{c_1^2} = 1$, and hence we have

$$\frac{x'^2}{a_1^2\left(1 + \frac{\theta_1}{a_1^2}\right)^2} + \frac{y'^2}{b_1^2\left(1 + \frac{\theta_1}{b_1^2}\right)^2} + \frac{z'^2}{c_1^2\left(1 + \frac{\theta_1}{c_1^2}\right)^2} = 1,$$

$$\text{or} \quad \frac{x'^2}{a_1^2 + 2\theta_1} + \frac{y'^2}{b_1^2 + 2\theta_1} + \frac{z'^2}{c_1^2 + 2\theta_1} = 1,$$

for the equation to the isothermal surface whose temperature is $v_1 + dv_1$, and which is therefore an ellipsoid described from the same foci as the original isothermal ellipsoid. In exactly the same manner it might be shown, that the isothermal surface whose temperature is $v_1 + dv_1 + dv_1'$, is an ellipsoid having the same foci as the ellipsoid whose temperature is $v_1 + dv_1$, and, consequently, as the original ellipsoid also. By continu-

ing this process it may be proved, that all the isothermal surfaces are ellipsoids, having the same foci as the original one.

From the form of the equation found above for the isothermal ellipsoid whose temperature is $v_1 + dv_1$, it follows, that θ_1 or $p_1\, dn_1$ is $= a_1\, da_1$, where da_1 is the increment of a_1, corresponding to the increment dn_1, of n_1. Hence, if a be one of the semiaxes of an ellipsoid, $a + da$ the corresponding semiaxis of another ellipsoid, having the same foci, dn the thickness at any point of the shell bounded by the two ellipsoids, and p the perpendicular from the centre to the plane touching either ellipsoid at the same point, we have

$$\frac{dn}{da} = \frac{a}{p} \ldots\ldots (b).$$

All that remains to be done is to find the temperature at the surface of any given ellipsoid, having the same foci as the original ellipsoid. For this purpose, let us first find the value of $-\dfrac{dv}{dn}$ at any point in the surface of the isothermal ellipsoid whose semiaxes are a, b, c. Now, we have, from (a),

$$-\frac{dv}{dn} = 4\pi kp,$$

where k is constant for any point in the surface of the isothermal ellipsoid under consideration, and determined by the condition, that the whole flux of heat across this surface must be equal to the whole flux across the surface of the original ellipsoid. Now the first of these quantities is equal to $4\pi k \iint p\, d\omega^2$, ($d\omega^2$ being an element of the surface) or to $4\pi \dfrac{ka}{\delta a} \iint \delta p d\omega^2$, since $\dfrac{\delta a}{a} = \dfrac{\delta p}{p}$. But $\iint \delta p d\omega^2$ is equal to the volume of a shell bounded by two similar ellipsoids, whose semiaxes are a, b, c, and $a + \delta a,\ b + \delta b,\ c + \delta c$, and is therefore readily shown to be equal to $4\pi \dfrac{\delta a}{a} abc$. Hence $4\pi \dfrac{ka}{\delta a} \iint \delta p d\omega^2$, or $4\pi k \iint p d\omega^2$ is equal to $4^2\pi^2 k\, abc$. In a similar manner we have, for the flux of heat across the original isothermal surface, $4^2\pi^2 k_1\, a_1 b_1 c_1$, and therefore

$$4^2\pi^2 kabc = 4^2\pi^2 k_1 a_1 b_1 c_1,$$

$$\text{which gives } k = k_1 \frac{a_1 b_1 c_1}{abc}.$$

Hence, we have

$$-\frac{dv}{dn} = 4\pi k_1 \frac{a_1 b_1 c_1}{abc} p \;\ldots\ldots\; (c).$$

The value of v may be found by integrating this equation. To effect this, since a, b, c are the semiaxes of an ellipsoid passing through the variable extremity of n, and having the same foci as the original ellipsoid, whose axes are a_1, b_1, c_1, we have

$$a^2 - a_1^2 = b^2 - b_1^2 = c^2 - c_1^2;$$

$$\left.\begin{aligned} \text{which gives} \quad b^2 &= a^2 - f^2 \\ c^2 &= a^2 - g^2 \\ \text{where} \quad f^2 = a_1^2 - b_1^2, \quad g^2 &= a_1^2 - c_1^2 \end{aligned}\right\} \ldots\ldots (d).$$

Hence (c) becomes

$$-\frac{dv}{dn} = 4\pi k_1 \frac{a_1 b_1 c_1 p}{a\sqrt{(a^2 - f^2)}\sqrt{(a^2 - g^2)}}.$$

Now, by (b), $dn = \dfrac{ada}{p}$, and hence

$$dv = -\,4\pi k_1 \frac{a_1 b_1 c_1\, da}{\sqrt{(a^2 - f^2)}\sqrt{(a^2 - g^2)}}.$$

Integrating this, we have

$$v = -\,4\pi k_1 a_1 b_1 c_1 \int \frac{da}{\sqrt{(a^2 - f^2)}\sqrt{(a^2 - g^2)}} + C \;\ldots.\; (e).$$

The two constants, k_1 and C, must be determined by the conditions $v = v_1$ when $a = a_1$, and $v = 0$ when $a = \infty$; the latter of which must be fulfilled, in order that the expression found for v may be equal to $\displaystyle\iint \frac{k_1 p_1 d\omega_1^2}{r_1}$.

To reduce the expression for v to an elliptic function, let us assume

$$\left.\begin{aligned} a &= f \operatorname{cosec} \phi \\ a_1 &= f \operatorname{cosec} \phi_1 \end{aligned}\right\} \ldots\ldots\ldots (f),$$

which we may do with propriety, if f be the greater of the two quantities f and g, since a is always greater than either of them, as we see from (d). On this assumption, equation (e) becomes

$$v = \frac{4\pi k_1 a_1 b_1 c_1}{f} \int_0^{\phi} \frac{d\phi}{\sqrt{(1 - c'^2 \sin^2 \phi)}} + C = \frac{4\pi k_1 a_1 b_1 c_1}{f} F_{c'}\phi + C$$

$$\text{where} \quad c' = \frac{g}{f} \ldots\ldots\ldots\ldots (g).$$

Determining from this the values of C and k_1, by the conditions mentioned above, we find $C = 0$, and

$$k_1 = \frac{fv_1}{4\pi a_1 b_1 c_1 F_{c'} \phi_1} \quad \ldots\ldots\ldots (h);$$

hence, the expression for v becomes

$$v = v_1 \frac{F_{c'} \phi}{F_{c'} \phi_1} \quad \ldots\ldots\ldots\ldots (k).$$

The results which have been obtained may be stated as follows:—

If, in an infinite solid, the surface of an ellipsoid be retained at a constant temperature, the temperature of any point in the solid will be the same as that of any other point in the surface of an ellipsoid described from the same foci, and passing through that point; and the flux of heat at any point in the surface of this ellipsoid will be proportional to the perpendicular from the centre to a plane touching it at the point, and inversely proportional to the volume of the ellipsoid.

This case of the uniform motion of heat was first solved by Lamé, in his Mémoire on Isothermal Surfaces, in Liouville's *Journal de Mathématiques*, Vol. II., p. 147, by showing, that a series of isothermal surfaces of the second order will satisfy the equation

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 0,$$

provided they are all described from the same foci. The value which he finds for v agrees with (e), and he finds, for the flux of heat at any point, the expression

$$\frac{KA}{\sqrt{(\mu^2 - \nu^2)}\sqrt{(\mu^2 - \rho^2)}};$$

or, according to the notation which we have employed,

$$\frac{4\pi k_1 a_1 b_1 c_1}{\sqrt{(a^2 - \nu^2)}\sqrt{(a^2 - \rho^2)}},$$

where ν is the greater real semiaxis of the hyperboloid of one sheet, and ρ the real semiaxis of the hyperboloid of two sheets, described from the same foci as the original ellipsoid, and passing through the point considered. Hence a^2, ν^2, ρ^2 are the three roots of the equation

$$\frac{x^2}{u} + \frac{y^2}{u - f^2} + \frac{z^2}{u - g^2} = 1,$$

or $u^3 - (f^2 + g^2 + x^2 + y^2 + z^2)\, u^2 + \{f^2g^2 + (f^2 + g^2)x^2 + g^2y^2 + f^2z^2\}\, u - f^2g^2x^2 = 0.$

Hence $a^2\nu^2\rho^2 = f^2g^2x^2$,

and $a^2\nu^2 + a^2\rho^2 + \nu^2\rho^2 = f^2g^2 + (f^2 + g^2)\,x^2 + g^2y^2 + f^2z^2$.

Therefore,

$$(a^2 - \nu^2)(a^2 - \rho^2) = a^4 - a^2\nu^2 - a^2\rho^2 - \nu^2\rho^2 + \frac{2a^2\nu^2\rho^2}{a^2}$$

$$= a^4 - \{f^2g^2 + (f^2 + g^2)\,x^2 + g^2y^2 + f^2z^2\} + 2\,f^2g^2\,\frac{x^2}{a^2}$$

$$= a^4 - (a^2 - b^2)(a^2 - c^2) - (2a^2 - b^2 - c^2)\,x^2 - (a^2 - c^2)\,y^2$$

$$- (a^2 - b^2)\,z^2 + 2\,(a^2 - b^2)(a^2 - c^2)\frac{x^2}{a^2}$$

$$= a^4 - (a^2 - b^2)(a^2 - c^2) - (b^2 + c^2)\,x^2 - (a^2 - c^2)\,y^2$$

$$- (a^2 - b^2)\,z^2 + 2\,b^2c^2\left(1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)$$

$$= a^4 - (a^2 - b^2)(a^2 - c^2) - (b^2 + c^2)\,x^2 - (a^2 + c^2)\,y^2 - (a^2 + b^2)\,z^2 + 2\,b^2c^2$$

$$= a^2b^2 + a^2c^2 + b^2c^2 - \{(b^2 + c^2)\,x^2 + (a^2 + c^2)\,y^2 + (a^2 + b^2)\,z^2\};$$

which is readily shown, by substituting for $a^2b^2 + a^2c^2 + b^2c^2$ its equal $(a^2b^2 + a^2c^2 + b^2c^2)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$, to be equal to $\frac{a^2b^2c^2}{p^2}$.

Hence the expression for $-\frac{dv}{dn}$, given above, becomes

$$-\frac{dv}{dn} = 4\pi k_1\,\frac{a_1b_1c_1}{abc}\,p,$$

which agrees with (c).

Attraction of a Homogeneous Ellipsoid on a Point within or without it.

If, in (c), we put $k_1 = \frac{da_1}{a_1}$, the value of $-\frac{dv}{dn}$ at any point will be the attraction on that point, of a shell bounded by two similar concentric ellipsoids, whose semiaxes are

$$a_1,\ a_1\sqrt{(1 - e^2)},\ a_1\sqrt{(1 - e'^2)},$$

$$\text{and}\quad a_1 + da_1,\ (a_1 + da_1)\sqrt{(1 - e^2)},\ (a_1 + da_1)\sqrt{(1 - e'^2)},$$

$$\left.\begin{aligned}\text{where}\quad a^2 - b^2 &= a_1^2 - b_1^2 = a_1^2e^2\\ \text{and}\quad a^2 - c^2 &= a_1^2 - c_1^2 = a_1^2e'^2\end{aligned}\right\}\ \ldots\ldots\ (1),$$

the density of the shell being unity. Now this attraction is in a normal drawn through the point attracted, to the surface of the ellipsoid whose semiaxes are a, b, c. If we call

α, β, γ, the angles which this normal makes with the co-ordinates x, y, z, of the point attracted, we have

$$\cos\alpha = \frac{\frac{x}{a^2}}{\sqrt{\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)}} = \frac{px}{a^2},$$

$$\text{and similarly,}\quad \cos\beta = \frac{py}{b^2},\ \cos\gamma = \frac{pz}{c^2}.$$

Hence, calling dA, dB, dC, the components of the attraction parallel to the axes of co-ordinates, we have, from (c),

$$\left.\begin{aligned} dA &= 4\pi x\,\frac{b_1c_1}{a^3bc}\,p^2da_1 \\ dB &= 4\pi y\,\frac{b_1c_1}{ab^3c}\,p^2da_1 \\ dC &= 4\pi z\,\frac{b_1c_1}{abc^3}\,p^2da_1 \end{aligned}\right\} \ldots\ldots (2).$$

The integrals of these expressions, between the limits $a_1 = 0$, and $a_1 = a_1'$, are the components of the attraction of an ellipsoid whose semiaxes are a_1', b_1', c_1', or a_1', $a_1'\sqrt{(1-e^2)}$, $a_1'\sqrt{(1-e'^2)}$, on the point (x, y, z). Now, by (1), we may express each of the quantities b, c, b_1, c_1, in terms of a and a_1, and the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1,\quad\text{or}\quad \frac{x^2}{a^2}+\frac{y^2}{a^2-e^2a_1^2}+\frac{z^2}{a^2-e'^2a_1^2} = 1 \ldots . (3),$$

enables us to express either of the quantities a, a_1, in terms of the other. The simplest way, however, to integrate equations (2), will be to express each in terms of a third quantity,

$$u = \frac{a_1}{a} \ldots\ldots\ldots (4).$$

Eliminating a from (3), by means of this quantity, we have

$$a_1^2 = u^2x^2 + \frac{y^2}{u^{-2}-e^2} + \frac{z^2}{u^{-2}-e'^2}.$$

$$\text{Hence}\ \ a_1da_1 = \left\{ux^2 + \frac{u^{-3}y^2}{(u^{-2}-e^2)^2} + \frac{u^{-3}z^2}{(u^{-2}-e'^2)^2}\right\}du$$

$$= \left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)a_1^4u^{-3}du = a_1^4p^{-2}u^{-3}du.$$

Also, from (4), we have $a = \frac{a_1}{u}$; from which we find, by (1), $b = \frac{a_1}{u}\sqrt{(1-e^2u^2)}$, $c = \frac{a_1}{u}\sqrt{(1-e'^2u^2)}$. By (1) also, $b_1 = a_1\sqrt{(1-e^2)}$, $c_1 = a_1\sqrt{(1-e'^2)}$. Making these substitutions in (2), and integrating, we have, calling a' the value of a, when $a_1 = a_1'$,

$$\left.\begin{aligned} A &= 4\pi x\sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^{\frac{a_1'}{a'}}\frac{u^2du}{\sqrt{(1-e^2u^2)}\sqrt{(1-e'^2u^2)}} \\ B &= 4\pi y\sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^{\frac{a_1'}{a'}}\frac{u^2du}{(1-e^2u^2)^{\frac{3}{2}}(1-e'^2u^2)^{\frac{1}{2}}} \\ C &= 4\pi z\sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^{\frac{a_1'}{a'}}\frac{u^2du}{(1-e^2u^2)^{\frac{1}{2}}(1-e'^2u^2)^{\frac{3}{2}}} \end{aligned}\right\}\ldots\ldots(5).$$

If the point attracted be within the ellipsoid, the attraction of all the similar concentric shells without the point will be nothing; and hence the superior limit of u will be the value of $\frac{a_1}{a}$ at the surface of an ellipsoid, similar to the given one, and passing through the point attracted.

Now, in this case, $a_1 = a$, since a is one of the semiaxes of an ellipsoid passing through the point attracted, and having the same foci as another ellipsoid (passing through the same point), whose corresponding semiaxis is a_1. Hence, for an interior point, we have

$$\left.\begin{aligned} A &= 4\pi x\sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^1\frac{u^2du}{\sqrt{(1-e^2u^2)}\sqrt{(1-e'^2u^2)}} \\ B &= 4\pi y\sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^1\frac{u^2du}{(1-e^2u^2)^{\frac{3}{2}}(1-e'^2u^2)^{\frac{1}{2}}} \\ C &= 4\pi z\sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^1\frac{u^2du}{(1-e^2u^2)^{\frac{1}{2}}(1-e'^2u^2)^{\frac{3}{2}}} \end{aligned}\right\}\ldots\ldots(6).$$

These are the known expressions for the attraction of an ellipsoid on a point within it. Equations (5) agree with the expressions given in the Supplement to Liv. v. of Pontécoulant's "*Théorie Analytique du Système du Monde,*" where they are found by direct integration, by a method discovered by Poisson. They may also be readily deduced from equations (6), by Ivory's Theorem. Or, on the other hand, by a

comparison of them, after reducing the limits of the integrals to 0 and 1, by substituting $\frac{a'_1}{a'}v$ for u, with equation (6), Ivory's Theorem may be readily demonstrated.

P. Q. R.

V.—ON THE LIMITS OF MACLAURIN'S THEOREM.

By A. Q. G. Craufurd, M.A. of Jesus College.

To express by a single term the series which remains after the first n terms of Maclaurin's series are taken.

Let $\overset{a}{\underset{n}{C}}u$ denote the coefficient of a^n in the development of a function of a which is represented by u.

Let $f(x)$ represent any function of x which is developable in a series of positive ascending powers of x; and, first, suppose the series to be finite, and to contain $m+1$ terms.

Then,

$$f(x) = \overset{a}{\underset{0}{C}}f(a) + x\overset{a}{\underset{1}{C}}f(a) + x^2\overset{a}{\underset{2}{C}}f(a) \ldots\ldots + x^n\overset{a}{\underset{n}{C}}f(a)$$
$$+ x^{n+1}\overset{a}{\underset{n+1}{C}}f(a) + x^{n+2}\overset{a}{\underset{n+2}{C}}f(a) \ldots\ldots x^m\overset{a}{\underset{m}{C}}f(a).$$

Now $\overset{a}{\underset{n+1}{C}}f(a) = \overset{a}{\underset{n}{C}}\frac{f(a)}{a}$, and $\overset{a}{\underset{n+p}{C}}f(a) = \overset{a}{\underset{n}{C}}\frac{f(a)}{a^p}$.

Therefore the second line of the series is equivalent to

$$x^{n+1}\left\{\overset{a}{\underset{n+1}{C}}f(a) + x\overset{a}{\underset{n+1}{C}}\frac{f(a)}{a} + x^2\overset{a}{\underset{n+1}{C}}\frac{f(a)}{a^2} \ldots\ldots x^{m-(n+1)}\overset{a}{\underset{n+1}{C}}\frac{f(a)}{a^{m-(n+1)}}\right\},$$

or

$$x^{n+1}\overset{a}{\underset{n+1}{C}}f(a)\left\{1 + \frac{x}{a} + \frac{x^2}{a^2} \ldots\ldots + \frac{x^{m-(n+1)}}{a^{m-(n+1)}}\right\}$$

$$= x^{n+1}\overset{a}{\underset{n+1}{C}}\left\{f(a)\frac{\frac{x^{m-n}}{a^{m-n}} - 1}{\frac{x}{a} - 1}\right\}$$

$$= x^{n+1}\overset{a}{\underset{n+1}{C}}\left(\frac{f(a)}{a^{m-(n+1)}} \cdot \frac{x^{m-n} - a^{m-n}}{x - a}\right)$$

$$= x^{n+1}\overset{a}{\underset{m}{C}}\left(f(a)\frac{x^{m-n} - a^{m-n}}{x - a}\right).$$

$$\text{But } \overset{a}{\underset{m}{C}}u = \frac{1}{1.2.\ldots m}\left(\frac{d^m u}{da^m}\right)_{a=0}:$$

consequently the last expression for the remainder is equivalent to

$$\frac{x^{n+1}}{1.2.\ldots m}\cdot\left(\frac{d^m}{da^m}\cdot fa\cdot\frac{x^{m-n}-a^{m-n}}{x-a}\right)_{a=0}.$$

If the development of $f(x)$ is infinite, the terms beyond the $(n+1)^{\text{th}}$ will form the series

$$x^{n+1}\left\{\underset{n+1}{C}f(a) + x\underset{n+1}{C}\frac{f(a)}{a} + x^2\underset{n+1}{C}\frac{f(a)}{a^2} + \&\text{c. to infinity}\right\}.$$

This series is equivalent to

$$x^{n+1}\overset{a}{\underset{n+1}{C}}f(a)\left\{1+\frac{x}{a}+\frac{x^2}{a^2}+\&\text{c. to infinity}\right\}$$

$$= x^{n+1}\overset{a}{\underset{n+1}{C}}\left\{\frac{f(a)}{1-\frac{x}{a}}\right\} = \frac{x^{n+1}}{1.2.\ldots(n+1)}\cdot\left\{\frac{d^{n+1}}{da^{n+1}}\cdot\frac{f(a)}{1-\frac{x}{a}}\right\}_{a=0}.$$

By means of this expression Maclaurin's theorem becomes

$$f(x) = \{f(a)\}_{a=0} + \frac{x}{1}\left\{\frac{df(a)}{da}\right\}_{a=0} + \frac{x^2}{1.2}\left\{\frac{d^2f(a)}{da^2}\right\}_{a=0}\ldots.$$

$$+\frac{x^n}{1.2..n}\left\{\frac{d^nf(a)}{da^n}\right\}_{a=0} + \frac{x^{n+1}}{1.2..(n+1)}\left\{\frac{d^{n+1}}{da^{n+1}}\cdot\frac{f(a)}{1-\frac{x}{a}}\right\}_{a=0} \quad ..(1).$$

In like manner we may sum any number of terms of Taylor's series.

For this purpose I observe, that the coefficient of h^n in the development of $f(x+h)$ is

$$\overset{a}{\underset{n}{C}}f(x+a) \text{ or } \overset{a}{\underset{0}{C}}\frac{f(x+a)}{a^n}.$$

Therefore

$$f(x+h) = \overset{a}{\underset{0}{C}}f(x+a) + h\overset{a}{\underset{0}{C}}\frac{f(x+a)}{a} + h^2\overset{a}{\underset{0}{C}}\frac{f(x+a)}{a^2} + \&\text{c.}$$

$$= \overset{a}{\underset{0}{C}}f(x+a)\left(1+\frac{h}{a}+\frac{h^2}{a^2}+\&\text{c.}\right)$$

Hence, the terms which follow the $(n+1)^{\text{th}}$ are,

$$\overset{a}{\underset{0}{C}}f(x+a)\left(\frac{h^{n+1}}{a^{n+1}}+\frac{h^{n+2}}{a^{n+2}}+\&\text{c.}\right)$$

$$= h^{n+1}\overset{a}{\underset{0}{C}}\frac{f(x+a)}{a^{n+1}}\left(1+\frac{h}{a}+\frac{h^2}{a^2}+\&\text{c.}\right):$$

$(m+1)$ terms of this last series are equivalent to

$$h^{n+1}\overset{a}{\underset{0}{C}}\left\{\frac{f(x+a)}{a^{n+1}}\,\frac{\dfrac{h^{m+1}}{a^{m+1}}-1}{\dfrac{h}{a}-1}\right\}$$

$$=h^{n+1}\overset{a}{\underset{0}{C}}\left(\frac{f(x+a)}{a^{m+n+1}}.\ \frac{h^{m+1}-a^{m+1}}{h-a}\right);$$

which is equivalent to

$$h^{n+1}\overset{a}{\underset{m+n+1}{C}}\left\{f(x+a)\,\frac{h^{m+1}-a^{m+1}}{h-a}\right\};$$

or, if you will,

$$\frac{h^{n+1}}{1.2.\ldots.(m+n+1)}\left(\frac{d^{m+n+1}}{da^{m+n+1}}\,.f(x+a)\,\frac{h^{m+1}-a^{m+1}}{h-a}\right)_{a=0}.$$

If the series is infinite, the terms which follow that affected with h^n are,

$$h^{n+1}\overset{a}{\underset{0}{C}}\frac{f(x+a)}{a^{n+1}}\left(1+\frac{h}{a}+\frac{h^2}{a^2}+\text{\&c. to infinity}\right)$$

$$=h^{n+1}\overset{a}{\underset{n+1}{C}}\left\{\frac{f(x+a)}{1-\dfrac{h}{a}}\right\}=\frac{h^{n+1}}{1.2.\ldots.(n+1)}\cdot\left\{\frac{d^{n+1}}{da^{n+1}}.\frac{f(x+a)}{1-\dfrac{h}{a}}\right\}_{a=0}.$$

Hence Taylor's theorem becomes

$$f(x+h)=f(x)+\frac{h}{1}\,\frac{df(x)}{dx}+\frac{h^2}{1.2}\,\frac{d^2f(x)}{dx^2}+\text{\&c.}$$

$$+\frac{h^n}{1.2.\ldots.n}\,\frac{d^nf(x)}{dx^n}+\frac{h^{n+1}}{1.2.\ldots.(n+1)}\left\{\frac{d^{n+1}}{da^{n+1}}\,.\frac{f(x+a)}{1-\dfrac{h}{a}}\right\}_{a=0}.$$

It is scarcely necessary to observe, that the same method which was employed to sum the Remainder of Maclaurin's series, is applicable to a series which contains negative as well as positive powers.

VI.—SOLUTION OF A PROBLEM IN ANALYTICAL GEOMETRY.

By J. Booth,

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The line which joins the points of intersection of two focal right lines, containing a given angle θ, with the conic section, envelopes two conic sections having their foci coincident with the focus of the given section; and if ε and ε' be the eccentricities of the loci, e of the given section, p and p' the parameters of the loci, P that of the given section, we shall have the following relations between the eccentricities and parameters of the three conic sections,

$$\varepsilon^2 + \varepsilon'^2 = e^2, \quad p^2 + p'^2 = P^2.$$

Let the equation of the given section be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2ex}{a} = \frac{b^2}{a^2} \ldots\ldots\ldots (1),$$

the origin being placed at the focus and the axes parallel to the principal axes of the section.

Let $(y'x')$, $(y''x'')$, be the co-ordinates of the points in which the sides of the given angle θ intersect the curve: the equation of the line passing through those points is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x') \ldots\ldots\ldots (2);$$

or if $y' = mx' \ldots (3)$, $y'' = m'x'' \ldots (4)$,
be the equations of the sides of the angle, we find, eliminating y', y'', between (2), (3), (4),

$$y - mx' = \frac{mx' - m'x''}{x' - x''}(x - x') \ldots (5).$$

Let ξ and v denote the reciprocals of the intercepts of the axes of X and Y by the right line (5); then

$$\frac{1}{x'} = mv + \xi \ldots (6), \qquad \frac{1}{x''} = m'v + \xi \ldots (7).$$

Now, eliminating (x', y') from the three equations (1), (3), (6), we find the quadratic equation

$$(a^2 - b^4v^2)\,m^2 - 2\,(b^2aev + b^4\xi v)\,m + b^2 - 2aeb^2\xi - b^4\xi^2 = 0 \ldots (8).$$

Now this is precisely the equation we should have found for m'; hence m and m' are the roots of (8), or

$$m + m' = \frac{b^2aev + b^4\xi v}{a^2 - b^4v^2}, \quad mm' = \frac{b^2 - 2ab^2e\xi - b^4\xi^2}{a^2 - b^4v^2};$$

$$\text{hence} \quad m - m' = \frac{2ab\,\{b^2(\xi^2+v^2)+2ae\xi-1\}^{\frac{1}{2}}}{a^2-b^4v^2}.$$

Let the quantity under the radical sign be called M; then

$$\tan\theta = \frac{m-m'}{1+mm'} = \frac{\pm 2ab\sqrt{(M)}}{a^2-b^2M};$$

or solving this quadratic equation, we find

$$M = \frac{a^2(1\pm\cos\theta)^2}{b^2\sin^2\theta},$$

or replacing for M its value, reducing and taking the lower sign, we find

$$\frac{b^4(\xi^2+v^2)}{\left(b^2+a^2\tan^2\dfrac{\theta}{2}\right)} + \frac{2b^2ae\,.\,\xi}{b^2+a^2\tan^2\dfrac{\theta}{2}} = 1 \;\ldots\ldots\; (9);$$

had we taken the upper sign, we should have found for the tangential equation of the locus

$$\frac{b^4(\xi^2+v^2)}{b^2+a^2\cot^2\dfrac{\theta}{2}} + \frac{2b^2ae\,.\,\xi}{b^2+a^2\cot^2\dfrac{\theta}{2}} = 1 \;\ldots\ldots\; (10).$$

Now, in these equations, as the coefficients of ξ and v are equal, the foci of these sections are at the origin, or coincide with the focus of the given section.

To determine the axes, &c. of these loci. The tangential equation of a conic section whose semiaxes and eccentricity are A, B, and ε, the origin of co-ordinates being at a focus and parallel to the axes of the section, is

$$B^2(\xi^2+v^2) + 2A\varepsilon\,.\,\xi = 1 \;\ldots\ldots\ldots\; (a).$$

Comparing this equation (a) with (9), we get

$$B^2 = \frac{b^4}{b^2+a^2\tan^2\dfrac{\theta}{2}}, \qquad A\varepsilon = \frac{b^2ae}{b^2+a^2\tan^2\dfrac{\theta}{2}};$$

$$\text{hence} \quad \varepsilon = e\cos\frac{\theta}{2}, \quad\text{and}\quad \frac{B^2}{A} = \frac{b^2}{a}\cos\frac{\theta}{2}, \quad\text{or}\quad p = P\cos\frac{\theta}{2}.$$

Had we taken the upper sign, we should have found

$$\varepsilon' = e\sin\frac{\theta}{2}, \qquad p' = P\sin\frac{\theta}{2};$$

$$\text{hence} \quad \varepsilon^2+\varepsilon'^2 = e^2, \qquad p^2+p'^2 = P^2:$$

when θ is a right angle, the two loci coincide.

Had any other point except one of the foci been chosen, we should have found for the locus a curve whose tangential equation would be of the fourth degree; the curve in this particular case separating into two distinct curves, each of which is a conic section.

Had the given section been an equilateral hyperbola, and θ a right angle, the locus would have been found a parabola.

When the given angle θ revolves round the centre instead of the focus, the tangential equation of the locus is

$$\{a^2b^2(\xi^2+v^2)-(a^2+b^2)\}^2 = 4a^2b^2\cot^2\frac{\theta}{2}\{a^2\xi^2+b^2v^2-1\}.$$

VII.—NOTE ON A CLASS OF FACTORIALS.

By D. F. Gregory, M.A. Fellow of Trinity College.

We owe to Vandermonde the interesting Theorem, that Binomial factorials of any order, in which the successive factors differ by a constant quantity, can be expanded in terms of the simple Monomial factorials according to the law of the expansion of Newton's Binomial Theorem. That is to say, that if we put

$$x(x-1)(x-2)\ldots(x-n+1) = x^{|n},$$

we shall have

$$(x+y)^{|n} = x^{|n} + nx^{|n-1}y + n\frac{(n-1)}{1.2}x^{|n-2}y^{|2} + \&c.$$

This proposition, which may be proved by various methods, is readily seen to depend on the fact that these factorials are subject to the laws of combination in virtue of which the Theorem of Newton, as applied to ordinary algebraical quantities, is true. And perhaps the Theorem of Vandermonde derives its chief value from its being one of the few examples which we have of the extension of Algebraical Theorems to operations not originally included in the demonstration. The other examples which are known, are the Theorems in the Differential Calculus and the Calculus of Finite differences, which are proved by the method of the separation of the symbols.

Between these however and the Theorem of Vandermonde, there is one marked point of distinction: for whereas in the former Theorems the operation which is subject to the index-operation is different from that which forms the staple of ordinary algebra, while the index-operation is always the same

viz. the operation of repetition, in the latter Theorem the base which is subject to the index-operation is or may be the same as that in ordinary algebra, while the index-operation is different. As any Theorem which will add to examples of this kind must contribute to extend our knowledge of the combination of symbols in the direction in which such an extension seems to be most important, I will offer no apology for occupying a page or two, in demonstrating that a Theorem similar to that of Vandermonde is true of a class of factorials different from that of which he has treated. The factorials to which I allude, are those which are met with in expanding the cosine or the sine of a multiple arc according to the powers of the cosine or sine of the arc itself. These factorials, which are of a somewhat remarkable form, have, like ordinary factorials, an analogy with powers, and the proposition of which I speak is an example of this analogy.

On referring to Vol. II. p. 129 of this Journal, the reader will find the following expressions for $\cos n\theta$ and $\sin n\theta$ in terms of $\sin\theta$ when n is an integer,

$$\cos n\theta = \cos n\pi \left\{1 - \frac{n^2}{1.2} v^2 + \frac{n^2(n^2-2^2)}{1.2.3.4} v^4 - \frac{n^2(n^2-2^2)(n^2-4^2)}{1.2.3.4.5.6} v^6 + \&c.\right\},$$

$$\sin n\theta = \cos (n-1)\pi \left\{nv - \frac{n(n^2-1^2)}{1.2.3} v^3 + \frac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5} v^5 - \&c.\right\},$$

v being written for $\sin\theta$.

Now to exhibit the analogy which the factorials, which are the coefficients of the various terms in these expressions, have with powers, let us represent them by a notation corresponding to that of ordinary factorials, and let us write

$$n = n_{|1},\ n^2 = n_{|2},\ n(n^2-1^2) = n_{|3},\ n^2(n^2-2^2) = n_{|4},\ \&c.;$$

and generally

$$n_{|r} = n^2(n^2-2^2)(n^2-4^2)(n^2-6^2)\,..\{n^2-(r-2)^2\}\,..r \text{ being even},$$
$$n_{|r} = n.(n^2-1^2)(n^2-3^2)\,........\{n^2-(r-2)^2\}\,..r \text{ being odd}.$$

This notation being employed, the preceding expressions may be written

$$\cos n\theta = (-)^n \left\{1 - n_{|2}\frac{v^2}{1.2} + n_{|4}\frac{v^4}{1.2.3.4} - n_{|6}\frac{v^6}{1.2.3.4.5.6} + \&c.\right\}..(1),$$

$$\sin n\theta = (-)^{n-1}\left\{n_{|1}v - n_{|3}\frac{v^3}{1.2.3} + n_{|5}\frac{v^5}{1.2.3.4.5} - \&c.\right\}.......(2).$$

Now the proposition which we have to demonstrate may be expressed, by means of this notation, in the following manner:

$$(m+n)_{|p} = m_{|p} + pm_{|p-1}\,n_{|1} + \frac{p(p-1)}{1.2}\,m_{|p-2}\,n_{|2} + \&c. + n_{|p}.$$

In the demonstration we must distinguish two cases according as p is even or odd.

1st. Let p be even and $=2r$; then putting $m+n$ instead of n in the series (1) we have

$$\cos(m+n)\theta=(-)^{m+n}\left\{1-(m+n)_{|2}\frac{v^2}{1.2}+\&c.+(-)^r(m+n)_{|2r}\frac{v^{2r}}{1.2..r}-\&c.\right\}..(3).$$

But by an ordinary formula of Trigonometry we have

$$\cos(m+n)\theta=\cos m\theta\cos n\theta-\sin m\theta\sin n\theta.$$

In this formula, if we substitute for the cosines and sines their equivalent series given in (1) and (2), and if we equate the coefficients of v^{2r}, and then multiply both sides of the equation by $1.2....2r$, we find

$$(m+n)_{|2r}=m_{|2r}+2r\,m_{|2r-1}n_{|1}+\frac{2r(2r-1)}{1.2}m_{|2r-2}n_{|2}+\&c.+n_{|2r},$$

which proves the theorem when p is even.

2nd. Let p be odd and $=2r+1$; then, by means of the series (2) and the formula $\sin(m+n)\theta=\sin m\theta\cos n\theta+\cos m\theta\sin n\theta$, we find on equating the coefficients of v^{2r+1}, and multiplying both sides of the equation by $1.2.3..(2r+1)$,

$$(m+n)_{|2r+1}=m_{|2r+1}+(2r+1)\,m_{|2r}n_{|1}+\frac{(2r+1)\,2r}{1.2}m_{|2r-2}n_{|2}+\&c.$$

which proves the theorem when p is odd.

It is easily seen that this result applies equally to factorials of the form

$$n^2(n^2-2^2h^2)(n^2-4^2h^2)....\{n^2-(r-2)^2h^2\},$$

since this last may be written under the form

$$h^r\frac{n^2}{h^2}\left(\frac{n^2}{h^2}-2^2\right)\left(\frac{n^2}{h^2}-4^2\right)....\left\{\frac{n^2}{h^2}-(r-2)^2\right\},$$

which, with the exception of the factor h^r, is the same in form as the factorials which we considered before.

I have not time at present to enter into any further developments of the nature of these factorials, and more particularly into the consideration of their interpretation when the index is negative or fractional: but this is of the less importance, as it is not very likely that expressions of this form will ever be extensively used in analysis; and the demonstration of the preceding theorem is given, not on account of its intrinsic value, but because it illustrates a part of the theory of Algebra which stands most in need of such examples.

VIII.—REMARKS ON THE DISTINCTION BETWEEN ALGEBRAICAL AND FUNCTIONAL EQUATIONS.

By R. L. Ellis, B.A. Fellow of Trinity College.

The distinction which it is usual to make between algebraical and functional equations, will not, I think, bear a strict examination. It is generally said that an algebraical equation determines the value of an unknown quantity, while a functional equation determines the form of an unknown function. But, in reality, the unknown quantity in the former case is a function of the coefficients of the equation, and our object in solving it is simply to ascertain the form of this function. Thus it appears, that in both cases the forms of functions are what we seek.

Let us therefore consider the subject in a more general manner, and endeavour to find a more decided point of distinction. The science of symbols is conversant with operations, and not with quantities; and an equation, of whatever species, may be defined to be a congeries of operations, known and unknown, equated to the symbol zero. Every operation implies the existence of a base, or something on which the operation is performed—in the language of Mr. Murphy, a subject. But the base of an operation is often the result of a preceding one. Thus, in $\log x^2$, the base of the operation log is x^2, itself the result of the operation expressed by the index on the base x. This in its turn may be considered as the result of an operation performed on the symbol unity. But in every kind of equation there is a point at which the farther analysis of symbols into operations on certain bases becomes irrelevant; and thus we are led in every case to recognize the existence of ultimate bases.

To solve an equation of any kind, is to determine the unknown operations by means of the known. If one symbol is said to be a function of another, it is, in reality, the result of an operation performed upon it. Thus the idea of functional dependence pervades the whole science of symbols, and on this idea the following remarks are based.

In order to classify equations, we can make use of two considerations: 1st. The nature of the operations which are combined together; 2nd. The order in which they succeed one another in the congeries of operations which is made equal to zero.

Let us illustrate these remarks by some examples.

If we have an equation of the form

$$x^2 + ax + b = 0 \;\ldots\ldots\ldots\; (1),$$

the bases are a and b; the operations are, first, the unknown one denoted by x, and then certain known ones denoted by the index, the coefficient, &c. All these are what are called algebraical operations.

If again we have an equation of the form

$$\frac{dy}{dx} - x = 0 \ldots\ldots\ldots (2),$$

the base is x; the operations are, first, the unknown one denoted by y, which is a function of x, then the operation $\frac{d}{dx}$, and lastly, certain algebraical operations. From the presence of the operation $\frac{d}{dx}$, this is called a differential equation. Equations (1) and (2) are discriminated by the nature of the operations combined, on our first principle of classification.

But in one important point these equations agree. In both, the unknown operation is performed immediately on the bases; the known are subsequent to the unknown: but in what are called functional equations this is not so. Thus, in the equation

$$\phi(mx) + x = 0 \ldots\ldots (3),$$

the base is x, the unknown operation is ϕ, which is performed, not on x, but on the result of a previous operation. In the preceding example the previous operation is known; but this is not essential. Thus in

$$\phi\phi x + x = 0 \ldots\ldots (4),$$

the previous operation denoted by the right-hand ϕ is unknown. The operation $\frac{d}{dx}$ may enter into equations where the unknown operation is not performed on the base. Thus we may have an equation of the form

$$\phi \frac{d}{dx} \phi x + x = 0 \ldots\ldots (5).$$

Equations (3), (4), (5), are functional equations; (3), (4), are ordinary functional equations; (5) is a differential functional equation; (3) is said to be of the first order, (4) of the second.

The introduction of the functional notation appears to be sometimes taken as the essence of functional equations; but if we wrote (1) and (2) thus,

$$\{\phi(ab)\}^2 + a\phi(ab) + b = 0 \ldots\ldots (1)',$$

$$\frac{d}{dx}\phi(x) - x = 0 \ldots\ldots (2)',$$

they would still be perfectly distinct from (3) or (4) or (5). The name functional equation is not happy; it refers to the notation, and not to the essence of the thing.

A question now arises: To what class shall we refer equations in finite differences? These are generally of the form

$$F(x, y_x, y_{x+1} \ldots .) = 0 \ldots \ldots (6),$$

where y_x is an unknown function, say $\phi(x)$ of x; so that (6) may be written thus,

$$F\{x, \phi x, \phi(x+1) \ldots .\} = 0.$$

Here the unknown operation is ϕ, which in the case of $\phi(x+1)$ is performed, not upon the base x, but on $x+1$. Thus it appears, that equations in finite differences are only a case of ordinary functional equations of the first order: and this is the reason why, in researches on functional equations, we perpetually meet with cases in which they may be reduced to equations in finite differences.

The preceding remarks contain, I think, the outline of a natural arrangement of the science of symbols. It is not difficult to overrate the importance of a mere classification; but I hope to be able to show, that the considerations now suggested are not without some degree of utility.

As the distinction between functional and common equations depends on the order of operations, it follows that, when part of the solution of an equation does not vary with the nature of the operation subjected to the resolving process, this part is applicable as much to functional equations as to any other. The special application of this principle to the discussion of a class of differential functional equations will be the object of a subsequent paper.

In the preceding remarks, operations of derivation, such as D, Δ, &c. are supposed to be replaced by functional operations in every case in which this can be effected.

IX.—MATHEMATICAL NOTES.

1. In the Examination Papers for 1834, the following problem is given: "If the chord of a conic section, whose eccentricity is e, subtend at its focus a constant angle $2a$, prove that it always touch a conic section having the same focus whose eccentricity is $e \cos a$." A solution of this problem by a peculiar analysis will be found in a preceding article; but the following method may be found not uninteresting.

Let r_1, r_2, be radii vectores to the ends of the chord, $\phi - a$, $\phi + a$, the corresponding angles vectores, p the perpendicular from the focus on the chord;

$$\therefore p \times \text{chord} = r_1 r_2 \sin 2a,$$

$$\therefore \frac{1}{p} = \frac{\sqrt{(r_1^2 + r_2^2 - 2r_1 r_2 \cos 2a)}}{r_1 r_2 \sin 2a} = \operatorname{cosec} 2a \sqrt{\left(\frac{1}{r_1^2} + \frac{1}{r_2^2} - \frac{2}{r_1 r_2} \cos 2a\right)},$$

$$\frac{1}{r_1} = \frac{1 + e \cos(\phi - a)}{l}, \quad \frac{1}{r_2} = \&c.$$

For $\cos 2a$ put $1 - 2\sin^2 a$; then, by a few obvious steps,

$$\frac{1}{p} = \frac{1}{l \cos a} \sqrt{(1 + 2e \cos a \cos \phi + e^2 \cos^2 a)}.$$

Put $a = 0$; then the chord becomes the tangent, and

$$\frac{1}{p_0} = \frac{1}{l} \sqrt{(1 + 2e \cos \phi + e^2)}.$$

But the general form coincides with this, if we put

$$l \cos a = \lambda \quad \text{and} \quad e \cos a = \varepsilon;$$

for then

$$\frac{1}{p} = \frac{1}{\lambda} \sqrt{(1 + 2\varepsilon \cos \phi + \varepsilon^2)}.$$

Hence p is *generally* the perpendicular on a tangent of an ellipse of eccentricity $e \cos a$. Hence the chord touches such an ellipse. The latus rectum is diminished in the same ratio as the eccentricity.

ε.

2. The relation between the long inequalities of two mutually disturbing planets, may be easily found without having recourse to the development of the disturbing function.

Let m, m', be the masses of the planets, a, a', the major axes of their orbits, n, n', their mean motions, h, h', twice the areas described in 1″; then we have

$$n = \frac{\mu^{\frac{1}{2}}}{a^{\frac{3}{2}}}, \quad n' = \frac{\mu^{\frac{1}{2}}}{a'^{\frac{3}{2}}},$$

μ being the mass of the Sun, in comparison with which the masses of the planets are neglected, so that it is the same for both. Taking the logarithmic differentials of these equations, and replacing the differentials by differences, we find

$$\frac{\Delta n}{n} = -\frac{3}{2} \frac{\Delta a}{a}, \quad \frac{\Delta n'}{n'} = -\frac{3}{2} \frac{\Delta a'}{a'}.$$

But by the principle of the conservation of areas,

$$mh + m'h' = \text{const.}$$

so that $$m\Delta h + m'\Delta h' = 0.$$

Now the orbits being supposed circular, we have

$$h = (\mu a)^{\frac{1}{2}}, \quad h' = (\mu a')^{\frac{1}{2}}:$$

hence $$\frac{\Delta h}{h} = \frac{1}{2}\frac{\Delta a}{a}, \quad \frac{\Delta h'}{h'} = \frac{1}{2}\frac{\Delta a'}{a'}:$$

Therefore we have

$$\frac{\Delta n}{n} : \frac{\Delta n'}{n'} = \frac{\Delta a}{\Delta a'} \cdot \frac{a'}{a} = \frac{\Delta h}{\Delta h'} \cdot \frac{h'}{h} = -\frac{m'}{m}\frac{a'^{\frac{1}{2}}}{a^{\frac{1}{2}}};$$

and $\dfrac{\Delta n}{n}$ and $\dfrac{\Delta n'}{n'}$ are the inequalities due to the disturbances, so that their ratio is thus given.

ε.

3. *Napier's Analogies.*—The form given by Professor Wallace to the mode of solving the spherical triangle whose sides are given, will probably be introduced into all future works on the subject. The corresponding mode of demonstrating Napier's Analogies should not be omitted.

$$M = \sqrt{\{\sin(s-a) \cdot \sin(s-b) \cdot \sin(s-c) \div \sin s\}},$$

$$\tan\tfrac{1}{2}A = \frac{M}{\sin(s-a)}, \quad \tan\tfrac{1}{2}B = \frac{M}{\sin(s-b)}, \quad \tan\tfrac{1}{2}C = \frac{M}{\sin(s-c)},$$

$$\tan\tfrac{1}{2}A \tan\tfrac{1}{2}B = \frac{M^2}{\sin(s-a) \cdot \sin(s-b)} = \frac{\sin(s-c)}{\sin s},$$

$$\frac{\tan\frac{1}{2}A \cdot \tan\frac{1}{2}C \pm \tan\frac{1}{2}B \cdot \tan\frac{1}{2}C}{1 \mp \tan\frac{1}{2}A \cdot \tan\frac{1}{2}B} = \frac{\sin(s-b) \pm \sin(s-a)}{\sin s \mp \sin(s-c)}$$

$$= \frac{\dfrac{\sin}{\cos}\dfrac{c}{2} \cdot \dfrac{\cos}{\sin}\dfrac{a-b}{2}}{\dfrac{\cos}{\sin}\dfrac{a+b}{2} \cdot \dfrac{\sin}{\cos}\dfrac{c}{2}}, \qquad (2s - a - b = c),$$

$$\text{or } \tan\frac{A \pm B}{2} \cdot \tan\frac{C}{2} = \frac{\dfrac{\cos}{\sin}\dfrac{a-b}{2}}{\dfrac{\cos}{\sin}\dfrac{a+b}{2}}.$$

A. D. M.

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I.—ON THE GENERAL PROBLEM OF INTERCALATIONS.

1. It having been so ordered, for beneficent ends, as that no one of the three most obvious measures of time is a multiple of another, the necessity of astronomical intercalations was discovered very early; and various methods have been used to provide against the equinoxes and solstices wandering inconveniently far from assigned points in the civil year. So long as this is all that is proposed, the problem of intercalations is plainly indeterminate, and we may choose such a solution as appears to furnish the most simple rule for determining the length of any proposed year. The Gregorian intercalation answers this purpose very well; but it does not* fulfil the condition, evidently possible, that the sun shall be in the vernal equinox, or any other particular point of his orbit, during the same assigned space of twenty-four hours in each year. It does not distribute the leap-years as evenly as possible; insomuch that if we select from a period of 400 years that part which contains the greatest proportion of leap-years, namely that from A.D. $(4m-1)\,100+3$ to $400m+96$, we find that the equinoxes and solstices happen 2 days and nearly 6 hours earlier in the latter year than in the former. It is true that this is of no great practical consequence, but still a mathematician may well like to know *what is the most perfect system of intercalation*. The problem is one *sui generis*, and its solution is given in a rather intricate, but at the same time elegant, law. It may be enunciated generally as follows.

* It is surprising that so eminent an astronomer as Sir John Herschel should have asserted the contrary. See Dr. Lardner's Cabinet Cyclopædia, Astronomy, p. 410.

2. Given two numbers whose common measure is not greater than unity, it is required to determine a series of multiples of the smaller number, whose excesses above the successive terms of an arithmetic progression whose common difference is the greater number, shall be the least possible.

3. Let a and b be the two numbers, of which a is the greater; and let x be any integer. We have to find an integer y, such that $by - (ax - f0)$ may be the least possible positive quantity, that is, that it may lie between 0 and b; $f0$ being arbitrary, but contained between the same limits.

4. We may assume

$$by - ax + f0 = fx; \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

then $fx > 0$, but $< b$. Taking the difference with respect to x,

$$b\Delta y - a = \Delta fx;$$

$$\text{therefore} \quad \Delta y = \frac{a + \Delta fx}{b}.$$

Now Δfx may be either positive or negative, but it must be arithmetically less than b. Let n be the whole number next less than $\frac{a}{b}$, then the preceding equation gives

$$\Delta y = n \text{ or } n + 1, \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\text{and} \quad \Delta fx = nb - a \text{ or } (n+1)b - a,$$

$$= -(a - nb) \text{ or } b - (a - nb) \quad \ldots\ldots\ldots (3).$$

5. Since Δfx is sometimes negative and sometimes positive, and fx always lies between 0 and b, its values must range continually between maxima and minima. Also, since one of the quantities $a - nb$, $b - (a - nb)$, must be greater than $\frac{1}{2}b$, that value of Δfx which, abstracted from its sign, is the greater, and the corresponding value of Δy, cannot exist for two successive values of x, or fx would transgress one of its limits. Hence, according as

$$a - nb > \text{ or } < b - (a - nb),$$

the minimum values of fx must immediately succeed the maximum, or conversely. Let z, $z + p$, be two successive values of x which render fx a minimum or maximum, accordingly; and let b' represent the least of the quantities $a - nb$, $b - (a - nb)$. Then while x increases from z to $z + p - 1$, $\Delta fx = \pm b'$; and for the next term, $\Delta fx = \mp (b - b')$. Hence

$$f(z+p) = fz \pm (p-1)b' \mp (b - b'),$$

$$\text{or} \quad f(z+p) - fz = \pm pb' \mp b \quad \ldots\ldots\ldots (4),$$

therefore $$p = \frac{b}{b'} \pm \frac{f(z+p) - fz}{b'} \ldots\ldots\ldots\ldots (5).$$

6. In order to determine p, it is requisite to find the limits of the maximum and minimum values of fx, which is done as follows. If fz be a minimum,

$$f(z-1) = fz + b - b',$$

and this must be $< b$, therefore

$$fz < b' \ldots\ldots\ldots\ldots\ldots\ldots (6);$$

again, if fz be a maximum,

$$f(z-1) = fz - (b - b'),$$

and this must be > 0, therefore

$$fz > b - b' \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

7. Thus, when fz and $f(z+p)$ are minima, they are each > 0 and $< b'$; and when they are maxima, they are each $> b - b'$ and $< b$. Therefore, in either case,

$$f(z+p) - fz > -b' \text{ and } < +b'.$$

Applying these limits to equation (5), it follows that

$$p > \frac{b}{b'} - 1, \text{ and } < \frac{b}{b'} + 1.$$

Let n' be the integer next less than $\frac{b}{b'}$. Then

$$p = n' \text{ or } n' + 1 \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

8. It remains to find when p has the one, and when the other, of these values. For this purpose, let x' be the number of values of z in the interval from 0 to x, so that while x increases from z to $z+p$, x' increases to $x'+1$; and let $f'x'$ be assumed

$$= fz \text{ or } b - fz \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

according as fz is a minimum or maximum value of fx; that is, according as $a - nb >$ or $< b - (a - nb)$. Then

$$\Delta f'x' = \pm \{f(z+p) - fz\},$$

which, by (4), $= pb' - b$.

Substituting the values of p from (8), we find

$$\Delta f'x' = -(b - n'b') \text{ or } b' - (b - n'b'),$$

equations exactly similar to those marked (3). Also we have, in either case, by (6), (7), and (9),

$$f'x' > 0, \text{ and } < b'.$$

9. Hence it follows, by the same reasoning as was before used in § 5, that according as $b - n'b' >$ or $< b' - (b - n'b')$, the value n' or $n' + 1$ of p cannot continue for two successive values of x', or, consequently, of z; and in order to carry the investigation further, we have only to change a, b, x, fx, n, in the preceding formulæ, into $b, b', x', f'x', n'$, and $z, p, b', n', x', f'x'$, into $z', p', b'', n'', x'', f''x''$, and so on. Thus we find, that the maximum and minimum values of fx will also have their maximum and minimum values, which we may call maxima and minima of the second order; and that every maximum and minimum of the second order corresponds to a change in the value of p. In like manner, every maximum and minimum of the third order corresponds to a change in the value of p', which represents the number of maxima or minima of the first order between two successive maxima or minima of the second order. By this method, though we cannot assign the successive values of y, *ad infinitum*, we may readily do so for a vast number of terms; since, by repeating r times the operations indicated above, we obtain the law of a number of successive terms greater than $nn'n''..n^{(r)}$, as will be more fully explained hereafter.

10. It is desirable to have an easy method of finding the numbers n, n', n'', &c. For this purpose let $\frac{a}{b}$ be developed in a continued fraction; then the first quotient will be n, but the others will not always be n', n'', &c.

Assume therefore $$\frac{a}{b} = n + \frac{1}{n_1 +}\,\frac{1}{n_2 +}\,\frac{1}{n_3 +}\cdots\cdot$$

then $$a - nb = \frac{b}{n_1 +}\,\frac{1}{n_2 +}\,\frac{1}{n_3 +}\cdots\cdot$$

and according as $a - nb >$ or $< b - (a - nb)$,

or $a - nb >$ or $< \frac{1}{2} b$,

we find $$\frac{1}{n_1 +}\,\frac{1}{n_2 +}\,\frac{1}{n_3 +}\cdots\cdot > \text{ or } < \tfrac{1}{2},$$

and $$n_1 + \frac{1}{n_2 +}\,\frac{1}{n_3 +}\cdots\cdot < \text{ or } > 2,$$

and therefore $n_1 =$ or > 1.

Hence, if $n_1 = 1$, we have, by § 5,

$$b' = b - (a - nb)$$
$$= b - \frac{b}{1 +}\,\frac{1}{n_2 +}\,\frac{1}{n_3 +}\cdots\cdot;$$

therefore $\dfrac{b}{b'} = \dfrac{1}{1-}\;\dfrac{1}{1+}\;\dfrac{1}{n_2+}\;\dfrac{1}{n_3+}\dots\dots$

$$\text{Let}\quad n_2 + \frac{1}{n_3+}\dots\dots = N,$$

$$\text{then}\quad \frac{b}{b'} = \frac{1}{1-}\;\frac{1}{1+}\;\frac{1}{N} = \frac{1}{1-}\;\frac{N}{N+1} = N+1,$$

$$= n_2 + 1 + \frac{1}{n_3+}\;\frac{1}{n_4+}\dots\dots;$$

therefore (§ 7) in this case $n' = n_2 + 1$.

But if n_1 be greater than unity,

$$b' = a - nb,$$

$$\text{and}\quad \frac{b}{b'} = n_1 + \frac{1}{n_2+}\;\frac{1}{n_3+}\dots\dots;$$

therefore in this case $n' = n_1$.

11. Hence we have the following rule for determining the numbers n, n', n'', &c.

Write down in order the successive quotients of the development of $\dfrac{a}{b}$ in a continued fraction. Whenever any quotient, other than the first, is unity, join it by addition to the following quotient. The series thus produced consists of the numbers required.

12. The law of the values of y may now be explained in any particular case. Suppose that n_1, n_2, n_3, &c. are all greater than unity. We have first, by (2), two values of Δy, namely n and $n+1$. The value $n+1$ can exist for one term only at a time (§ 5), but the value n, by equation (8), continues for either $n'-1$ or n' terms. Hence we have

First, a period of n' terms,

Secondly, a period of $n'+1$ terms,

both of which are terminated by a value of Δy equal to $n+1$.

Let these be called the smaller and greater periods of the first order.

Again we have two periods of the second order; the smaller containing $n''-1$ smaller periods of the first order, and one greater period of the first order; the greater containing n'' smaller periods of the first order, and one greater period of the first order.

Similarly the smaller period of the third order will contain $n'''-1$ smaller periods of the second order, and one greater

period of the second order; and the greater period of the third order will contain n''' smaller periods of the second order, and one greater period of the second order.

But if $n_1 = 1$, the periods of the first order will be terminated by a value of Δy equal to n, that function having been equal to $n+1$ throughout the rest of the period. And if $n^{(r)}$ result from the addition of two consecutive quotients, of which the former is equal to unity, the smaller period of the r^{th} order will contain $n^{(r)} - 1$ *greater* periods of the $(r-1)^{\text{th}}$ order, and one *smaller* period of that order; and the greater period of the r^{th} order will contain $n^{(r)}$ *greater* periods of the $(r-1)^{\text{th}}$ order, and one *smaller* period of that order.

13. To complete the investigation, it remains that, knowing the value of $f0$ in (1), we should find the distance of the first terms of the periods of the several orders from the beginning of the series of the values of y, that is, that we should find the lowest values of z, z', &c.

We have then, according as $a - nb >$ or $< b - (a - nb)$, (§. 5), that is, according as $n =$ or > 1 (§ 10),

$$\Delta fx = \pm b'$$

from $x = 0$ till fx becomes $> b - b'$, or $< b'$. See (6) and (7). Hence if z_0 represent the lowest value of z,

$$fz_0 = f0 \pm (z_0 - 1)\, b' \mp (b - b')$$
$$= f0 \pm z_0 b' \mp b \quad \ldots\ldots\ldots\ldots\ldots (10.)$$

Therefore $$z_0 = \frac{b}{b'} \pm \frac{fz_0 - f0}{b'}.$$

When the upper sign is to be taken,

$$fz_0 > 0, \text{ but } < b';$$

therefore $$z_0 > \frac{b - f0}{b'}, \text{ but } < \frac{b - f0}{b'} + 1;$$

therefore $z_0 =$ the integer next greater than $\dfrac{b - f0}{b'}$.

When the lower sign is to be taken,

$$fz_0 > b - b', \text{ but } < b;$$

therefore $$z_0 > \frac{f0}{b'}, \text{ but } < \frac{f0}{b'} + 1;$$

therefore $z_0 =$ the integer next greater than $\dfrac{f0}{b'}$.

If, in the first case, $f0$ should be less than b'; or, in the second, it should be greater than $b - b'$, the above formulæ will give n' or $n' + 1$ for the value of z_0. But the proper value of z_0 will on these suppositions be 0.

Having found z_0, fz_0 is known from equation (10); and we have next to find the place where the first period of the second order begins, or the value of z'_0. This is done in the same manner; only putting an additional accent on all the letters. $f'0$ is found from fz_0 by equation (9).

In the same manner the places where the periods of the third and higher orders begin, may be found.

14. As an example, let us take the question referred to in the beginning of this paper, that of determining the law of the length of the civil year, so that the place of the equinox may never vary twenty-four hours.

The ratio of the mean tropical year to the mean solar day has been found to be 365·242264. Let this be converted into a continued fraction, and we have the quotients

365, 4, 7, 1, 4, 1, 5, 1, &c.

Hence (§. 11),

365, 4, 7, 5, 6, &c.

are the respective values of

n, n', n'', n''', n'''', &c.

Following the principles explained in §. 12, we find for the two periods of the first order,

A period of 4 years, containing 3 of 365 days, and 1 of 366;

A period of 5 years, containing 4 of 365 days, and 1 of 366.

For the periods of the second order,

A period of 29 years, containing 6 periods of 4 years and 1 of 5.

A period of 33 years, containing 7 periods of 4 years and 1 of 5.

For the periods of the third order,

A period of 161 years, containing 4 periods of 33 years and 1 of 29;

A period of 194 years, containing 5 periods of 33 years and 1 of 29.

For the periods of the fourth order,

A period of 1131 years, containing 5 periods of 194 years and 1 of 161;

A period of 1325 years, containing 6 periods of 194 years and 1 of 161.

15. Our knowledge of the length of the tropical year is hardly exact enough to enable us to carry the series further, if it were of any use to do so. To accomplish the purpose

fully, it would be necessary to take into account the variation of the tropical year, which might probably be done without much difficulty. The system of intercalation given above agrees with that adopted by the Persians, as far as the period of 161 years. See the *History of Astronomy* in the Library of Useful Knowledge, chap. x.

16. In order to determine what place a given year has in any period, the limits between which an equinox is always to fall, must be assigned. If it be required that the vernal equinox shall always happen on the twenty-first of March, in a given longitude, it is to be observed that according to the original assumption, $f0$ represents the interval between the time of the sun passing the equinoctial point and the following midnight; so that this being known accurately in any given year, the beginnings of the several periods may be found by the method explained in §. 13. To this end we have

$$\begin{array}{ll} a = & 365{\cdot}242264 \\ b = & 1{\cdot}000000 \\ b' = a - nb = & {\cdot}242264 \\ b'' = b - n'b' = & {\cdot}030944 \\ b''' = (n'' + 1)\, b'' - b' = & {\cdot}005288 \\ b'' = (n''' + 1)\, b''' - b'' = & {\cdot}000784 \end{array}$$

The reader who is interested in this subject, may apply the preceding formulæ to the measurement of time by years and lunar months, or lunar months and days. Other problems, besides astronomical, may also be solved by means of them; for instance, that of representing, as correctly as possible, an oblique line by stitches on canvass; and that of building a wall, the top of which shall follow a given slope, with horizontal courses of brick.

S. S. G.

II.—NOTE ON THE THEORY OF THE SOLUTIONS OF CUBIC AND BIQUADRATIC EQUATIONS.

By James Cockle, B.A. Trinity College.

In the concluding section of Hymers' *Theory of Equations*, (1st Edit.) an outline is given of the method by which Lagrange (in the *Berlin Memoirs* for 1770,) succeeded in showing that all the particular and apparently isolated solutions of equations then known, were capable of being referred to one

general principle. The great interest attaching to the investigation induces me to hope, that the following extension of it, to one or two later cases, may not be unacceptable to the readers of this periodical.

Resuming then the equation (2), p. 249, vol. II. of the Journal, and multiplying both sides of it by the denominator of the right-hand side,

$$\{(n\rho)^{\frac{1}{3}} - \rho\} x = \{(n\rho)^{\frac{1}{3}} + a\} z + b$$

where $\rho = 3z + a$. Now let x_1, x_2, x_3, be the three values of x, and $\sqrt[3]{(n\rho)}$, $a\sqrt[3]{(n\rho)}$, $a^2\sqrt[3]{(n\rho)}$, the corresponding values of $(n\rho)^{\frac{1}{3}}$, then we have the following equations:

$$\{\sqrt[3]{(n\rho)} - \rho\} x_1 = \{\sqrt[3]{(n\rho)} + a\} z + b$$
$$\{a . \sqrt[3]{(n\rho)} - \rho\} x_2 = \{a . \sqrt[3]{(n\rho)} + a\} z + b$$
$$\{a^2 . \sqrt[3]{(n\rho)} - \rho\} x_3 = \{a^2 . \sqrt[3]{(n\rho)} + a\} z + b.$$

Add these equations, then, since $x_1 + x_2 + x_3 = -a$, and $1 + a + a^2 = 0$, we have

$$\sqrt[3]{(n\rho)}(x_1 + ax_2 + a^2x_3) + \rho a = 3az + 3b,$$

$\therefore$ substituting, transposing, and making $x_1 + ax_2 + a^2x_3 = Y$;

$$\sqrt{\{(a^2 - 3b)(3z + a)\}} . Y = -(a^2 - 3b),$$

$$\therefore z = -\frac{a}{3} - \frac{(a^2 - 3b)^2}{3Y^3} \quad \ldots\ldots\ldots\ldots (1).$$

Hence this solution, like others, is effected by forming a "reducing equation," whose roots are functions of Y^3, which quantity has (Hymers, pp. 189—192) only two values. If in the expression for x we write for z its value derived from (1), a simple and obvious reduction will give us the same values of x as are obtained at p. 192 of Hymers; and the fact of only one of the values of Y^3 entering into the expression for z, confirms my concluding remark in the last number of this publication.

On examining next the discussion of a biquadratic, (Hymers, p. 192), we see that there are three several systems of functions of the roots of the original equation, which, possessing only three values, are competent to form the roots of the "reducing equation;" these functions are of the forms

$$(x_1 + x_3)(x_2 + x_4), \quad x_1x_3 + x_2x_4, \quad \text{and} \quad (x_1 - x_2 + x_3 - x_4)^2,$$

which last is the square of y, and forms the roots of the reducing equation in Euler's method; the second applies to Waring's, and the first to the accompanying one, whose final cubic (provided the elimination be performed as below) is *essentially* different from the ordinary ones and coincides with

that in Hymers', p. 192, line 6 from the bottom; to which indeed it is the corresponding particular solution.

Besides the above, there is another more complicated function, having only three values, which forms the basis of the solution given by your correspondent ff; it is of the form $\dfrac{x_1x_3 - x_2x_4}{x_1 - x_2 + x_3 - x_4}$, as he has shown in vol. I. of the Journal.

Now, let $x^4 + px^3 + qx^2 + rx + s = 0$

be supposed to be made up of the factors

$$(x^2 + ex + f)(x^2 + gx + h);$$

then, multiplying and equating coefficients of like powers of x,

$$e + g = p \ldots\ldots (1), \quad f + h + eg = q \ldots\ldots (2),$$
$$fg + eh = r \ldots\ldots (3), \qquad fh = s \ldots\ldots (4);$$

but, $e - g = \surd\{(e+g)^2 - 4eg\} = \surd(p^2 - 4eg)$ by (1),

$f - h = \surd\{(f+h)^2 - 4fk\} = \surd\{(q - eg)^2 - 4s\}$ by (2) and (4);

and since (3) may be put under the form

$$r = (f + h)\frac{e+g}{2} - (f - h)\frac{e-g}{2};$$

and by substitution and transposition,

$$\surd(p^2 - 4eg)\,\surd\{(q - eg)^2 - 4s\} = p(q - eg) - 2r;$$

therefore squaring, transposing, and making $eg = z$,

$$z^3 - 2qz^2 + (pr + q^2 - 4s)z + p^2s - pqr + r^2 = 0 \ldots\ldots (2).$$

Had we made $f + h = z$, we should have had Waring's solution, or if we had taken $e - g$, we should have had the resulting cubic given in pp. 166 and 170 of Hymers, which is the basis of Euler's method; the process of elimination in each of these cases being of course slightly different.

Middle Temple, December 23, 1841.

III.—EXPOSITION OF A GENERAL THEORY OF LINEAR TRANSFORMATIONS.—PART II.

By George Boole.

From the great length to which the investigations of Part I. have extended, I shall in this paper chiefly confine myself to the exhibition of results, and shall leave it to the reader to supply the demonstrations omitted.

1. Let us represent the binary system of equations, $q = r$, and $Q = R$, under the forms

$$\Sigma k a_1 x_1^{\alpha} x_2^{\beta} \ldots x_m^{\mu} = \Sigma k b_1 y_1^{\alpha} y_2^{\beta} \ldots y_m^{\mu} \ldots\ldots\ldots (1),$$

$$\Sigma k A_1 x_1^{\alpha} x_2^{\beta} \ldots x_m^{\mu} = \Sigma k B_1 y_1^{\alpha} y_2^{\beta} \ldots y_m^{\mu} \ldots\ldots\ldots (2),$$

wherein a_1, b_1, A_1, B_1, are independent constants, susceptible each of a different value for every successive term of the function to which it belongs, k a numerical constant determined by the formula

$$k = \frac{1.2.3.\ldots n}{1.2.\ldots\alpha.1.2.\ldots\beta.\ldots 1.2.\ldots\mu},$$

and $\alpha, \beta, \ldots \mu$, indeterminate positive integers, (the value 0 included,) subject to the condition of homogeneity,

$$\alpha + \beta \ldots + \mu = n.$$

The relations among the constants of (1) and (2) derived from the dependent conditions, $\theta(Q + hq) = 0$, $\theta(R + hr) = 0$, may then be expressed under the symbolical forms,

$$\frac{\left(\Sigma a_1 \frac{d}{dA_1}\right)^{\eta} \theta(Q)}{\theta(Q)} = \frac{\left(\Sigma b_1 \frac{d}{dB_1}\right)^{\eta} \theta(R)}{\theta(R)} \ldots (3);$$

the values of η varying from 1 to γ the index of the degree of $\theta(Q)$. To elucidate more fully this notation, the above theorems may be compared with their equivalents in Part I., it being observed that $\dfrac{\theta(Q)}{\theta(q)} = \dfrac{\theta(R)}{\theta(r)}$.

2. The above results may be included in a very important generalization. Suppose that we have $\rho + 1$ equations, expressed under the general forms

$$\left.\begin{array}{lll} \Sigma k a_1 x_1^{\alpha} x_2^{\beta} \ldots x_m^{\mu} = \Sigma k b_1 y_1^{\alpha} y_2^{\beta} \ldots y_m^{\mu} \ldots & \text{for} & q_1 = r_1 \\ \ldots \quad \ldots \quad \ldots \quad \ldots & & \ldots \\ \Sigma k a_\rho x_1^{\alpha} x_2^{\beta} \ldots x_m^{\mu} = \Sigma k b_\rho y_1^{\alpha} y_2^{\beta} \ldots y_m^{\mu} \ldots & & q_\rho = r_\rho \\ \Sigma k A x_1^{\alpha} x_2^{\beta} \ldots x_m^{\mu} = \Sigma k B y_1^{\alpha} y_2^{\beta} \ldots y_m^{\mu} \ldots & & Q = R \end{array}\right\} \ldots (4);$$

then are the equations

$$\theta(Q + h_1 q_1 + h_2 q_2 \ldots + h_\rho q_\rho) = 0. \quad \theta(R + h_1 r_1 + h_2 r_2 \ldots + h_\rho r_\rho) = 0 \ldots (5),$$

identical relatively to $h_1, h_2, \ldots h_\rho$; and if we put

$$\Sigma\left(a_1 \frac{d}{dA}\right) = \phi_1, \quad \Sigma\left(b_1 \frac{d}{dB}\right) = \psi_1,$$

$$\Sigma\left(a_\rho \frac{d}{dA}\right) = \phi_\rho, \quad \Sigma\left(b_\rho \frac{d}{dB}\right) = \psi_\rho;$$

then are all the relations among the constants of (4) included in the theorem

$$\frac{\phi_1^a\phi_2^b \ldots\ldots \phi_\rho^r\ \theta(Q)}{\theta(Q)} = \frac{\psi_1^a\psi_2^b \ldots\ldots \psi_\rho^r\ \theta(R)}{\theta(R)} \ldots\ldots (6),$$

$a, b, \ldots\ldots r$, being indeterminate positive integers, the value 0 included, subject only to the condition that their sum shall not exceed γ. In place of (6), we may with somewhat greater generality say,

$$\frac{F(\phi_1\phi_2 \ldots\ldots \phi_\rho)\ \theta(Q)}{\theta(Q)} = \frac{F(\psi_1\psi_2 \ldots\ldots \psi_\rho)\ \theta(R)}{\theta(R)} \ldots\ldots (7);$$

F denoting any rational and interpretable combination of the symbols to which it is affixed.

(1) Let us take the ternary system of equations,

$$\left.\begin{aligned} ax^2 + by^2 + cz^2 + 2dyz + 2exz + 2fxy &= a'x'^2 + \&c. \\ a_1x^2 + b_1y^2 + c_1z^2 + 2d_1yz + 2e_1xz + 2f_1xy &= a'_1x'^2 + \&c. \\ a_2x^2 + b_2y^2 + c_2z^2 + 2d_2yz + 2e_2xz + 2f_2xy &= a'_2x'^2 + \&c. \end{aligned}\right\} \ldots\ldots (8);$$

the first of which, $q = r$, we shall put in place of $Q = R$, of (4), then

$$\phi_1 = a_1\frac{d}{da} + b_1\frac{d}{db} + \ldots\ldots + f_1\frac{d}{df} \quad \psi_1 = a'_1\frac{d}{da'} + \&c.$$

$$\phi_2 = a_2\frac{d}{da} + b_2\frac{d}{db} + \ldots\ldots + f_2\frac{d}{df} \quad \psi_2 = a'_2\frac{d}{da'} + \&c.$$

$$\theta(q) = abc + 2def - (ad^2 + be^2 + cf^2) \quad \theta(r) = a'b'c' + \&c.$$

As $\theta(q)$, $\theta(r)$ are of the third degree, it is manifest that all the forms deducible from (6) will be included in the following nine equations,

$$\frac{\phi_1\theta(q)}{\theta(q)} = \frac{\psi_1\theta(r)}{\theta(r)}, \quad \frac{\phi_2\theta(q)}{\theta(q)} = \frac{\psi_2\theta(r)}{\theta(r)},$$

$$\frac{\phi_1^2\theta(q)}{\theta(q)} = \frac{\psi_1^2\theta(r)}{\theta(r)}, \quad \frac{\phi_2^2\theta(q)}{\theta(q)} = \frac{\psi_2^2\theta(r)}{\theta(r)},$$

$$\frac{\phi_1\phi_2\theta(q)}{\theta(q)} = \frac{\psi_1\psi_2\theta(r)}{\theta(r)} \ldots\ldots (9),$$

$$\frac{\phi_1^2\phi_2\theta(q)}{\theta(q)} = \frac{\psi_1^2\psi_2\theta(r)}{\theta(r)}, \quad \frac{\phi_1\phi_2^2\theta(q)}{\theta(q)} = \frac{\psi_1\psi_2^2\theta(r)}{\theta(r)},$$

$$\frac{\phi_1^3\theta(q)}{\theta(q)} = \frac{\psi_1^3\theta(r)}{\theta(r)}, \quad \frac{\phi_2^3\theta(q)}{\theta(q)} = \frac{\psi_2^3\theta(r)}{\theta(r)}.$$

Of the above equations, the fifth, (9), will after development involve in its numerators the coefficients of all the

equations of the original system. By this circumstance it is distinguished from the remaining eight, which are in fact no other than would be obtained, by applying the theorem for binary systems to the separate pairs of equations which may be selected from (8). The development of (9) gives

$$\frac{\Sigma\{abc+2def-(ad^2+be^2+cf^2)\}}{abc+2def-(ad^2+be^2+cf^2)}=\frac{\Sigma(a'b'c'+\&c.)}{abc+\&c.},$$

where Σ, applied to any particular terms, denotes the aggregate of all similar combinations, which are of the first degree with respect to the coefficients of each of the primitive equations; thus,

$$\Sigma abc = ab_1c_2 + ab_2c_1 + a_1bc_2 + a_1b_2c + a_2bc_1 + a_2b_1c,$$

the remaining eight equations may be derived from (56), &c., Part I.

3. When the system (4) is linear, the theorem (6) ceases to be interpretable, and is replaced by the following. Let the values of $x_1, x_2, \dots x_m$, in terms of the other set of variables, be

$$\left.\begin{array}{l} x_1 = \lambda_1 y_1 + \lambda_2 y_2 \;\dots\; + \lambda_m y_m \\ x_2 = \mu_1 y_1 + \mu_2 y_2 \;\dots\; + \mu_m y_m \\ \quad\dots \qquad \dots \qquad \dots \\ x_m = \rho_1 y_1 + \rho_2 y_2 \;\dots\; + \rho_m y_m \end{array}\right\} \;\dots\dots\; (10);$$

and let E be that function of the constants obtained by eliminating the variables from the second members of the above equations, expressed under their most general forms, and equated to 0. Let E_1 be a similar function of the constants involved in the first members of any m equations of the original system (4), and E_2 of those entering into the second members of the same equations, then

$$EE_1 = E_2 \;\dots\dots\; (11),$$

whence all the relations may be found.

4. Hitherto we have confined ourselves to the supposition, that the equations of a proposed system are all of the same degree. When such is not the case, recourse must be had to one of the following methods:—

First, We may raise by involution each of the primitive equations to the degree indicated by the least common multiple of the indices of their degrees, and then apply the theorems of the preceding sections.

Secondly, The second method depends on the judicious application of the principle which I am about to demonstrate; a principle the applications of which are not restricted to homogeneous functions, and which, if properly

followed out, may be found of service in more than one department of analysis.

Let U, a function of $x_1, x_2, \ldots. x_m$, be linearly transformed into V, a function of $y_1, y_2, \ldots. y_m$, by virtue of the relations (10).

Differentiating, we get

$$\left.\begin{array}{l} dx_1 = \lambda_1 dy_1 + \lambda_2 dy_2 \ldots + \lambda_m dy_m \\ dx_2 = \mu_1 dy_1 + \mu_2 dy_2 \ldots + \mu_m dy_m \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ dx_m = \rho_1 dy_1 + \rho_2 dy_2 \ldots + \rho_m dy_m \end{array}\right\} \ldots\ldots (12),$$

$$\frac{dU}{dx_1}dx_1 + \frac{dU}{dx_2}dx_2 \ldots + \frac{dU}{dx_m}dx_m = \frac{dV}{dy_1}dy_1 + \frac{dV}{dy_2}dy_2 \ldots + \frac{dV}{dy_m}dy_m \ldots (13),$$

$$\frac{d^2U}{dx_1^2}dx_1^2 + 2\frac{d^2U}{dx_1dx_2}dx_1dx_2 + \&c. = \frac{d^2V}{dy_1^2}dy_1^2 + \frac{d^2V}{dy_1dy_2}dy_1dy_2 + \&c. \ldots (14).$$

$$\Sigma K \frac{d^nU}{dx_1^a dx_2^b \ldots dx_m^c} dx_1^a dx_2^b \ldots dx_m^c$$

$$= \Sigma K \frac{d^nV}{dy_1^a dy_2^b \ldots dy_m^c} dy_1^a dy_2^b \ldots dy_m^c \ldots. (15).$$

Now since $x_1, x_2, \ldots x_m$, are independent, $dx_1, dx_2, \ldots dx_m$, are independent also. The only relations into which they enter are those of (12), by which they are linearly connected with $dy_1, dy_2, \ldots dy_m$, in precisely the same way as $x_1, x_2, \ldots x_m$, are connected with $y_1, y_2, \ldots y_m$. It is hence evident, that the second members of (13), (14), (15), may be regarded as formed from their respective first members, by the substitution of the values of $dx_1, dx_2, \ldots dx_m$, given in (12). Indeed the coefficients, $\frac{dU}{dx}$, $\frac{dU}{dx_2}$, &c. though variable as being functions of $x_1, x_2, \ldots x_m$, are nevertheless constant relatively to $dx_1, dx_2, \ldots, dx_m$. It is therefore clear that the equations (13), (14), (15), regarded as homogeneous with respect to the differentials, fulfil among their coefficients the same relations, and are subject to the same general laws as if those coefficients were absolutely constant.

By the application of this principle, it may be shewn that *the discussion of a given multiple system of equations may be reduced to that of another system, whose common index shall be equal to the greatest common measure of the indices of the original equations, or to any proposed multiple of that quantity.*

5. Given the binary system,

$$ax + by = a'x' + b'y' \ldots. (16), \qquad q = r \ldots. (17),$$

the latter equation being homogeneous, and of the n^{th} degree.

By differentiation,

$$\left.\begin{aligned} adx + bdy &= a'dx' + b'dy' \\ \frac{dq}{dx}dx + \frac{dq}{dy}dy &= \frac{dr}{dx'}dx' + \frac{dr}{dy'}dy' \end{aligned}\right\} \ldots\ldots (18);$$

these equations being linear with respect to dx, dy, dx', dy', we have by (11),

$$E\left(a\frac{dq}{dy} - b\frac{dq}{dx}\right) = a'\frac{dr}{dy'} - b'\frac{dr}{dx'},$$

which may be written under the form,

$$E\left(a\frac{d}{dy} - b\frac{d}{dx}\right)q = \left(a'\frac{d}{dy'} - b'\frac{d}{dx'}\right)r,$$

and would give, if the differentiations were effected, a homogeneous equation of the $(n-1)^{\text{th}}$ degree. This we shall represent by $q' = r'$. Again, therefore, applying the theorem (11), we obtain

$$E\left(a\frac{d}{dy} - b\frac{d}{dx}\right)q' = \left(a'\frac{d}{dy'} - b'\frac{d}{dx'}\right)r',$$

or substituting for q' and r', the respective members for which they stand,

$$E^2\left(a\frac{d}{dy} - b\frac{d}{dx}\right)^2 q = \left(a'\frac{d}{dy'} - b'\frac{d}{dx'}\right)^2 r,$$

and thus finally, after η repetitions of the process,

$$E^\eta\left(a\frac{d}{dy} - b\frac{d}{dx}\right)^\eta q = \left(a'\frac{d}{dy'} - b'\frac{d}{dx'}\right)^\eta r \ldots\ldots (19).$$

It remains to determine E. Now by (87), Part I.,

$$\theta(q) = \frac{\theta(r)}{E^{\frac{\gamma n}{m}}} = \frac{\theta(r)}{E^{n(n-1)}} \ldots\ldots (20),$$

for when $m = 2$, it is easily seen that $\gamma = 2(n-1)$. And since the relations among the variables are the same as those among the differentials, it is evident that the value of E, determined from (20), may be applied in the present case. Substituting that value in (19), we obtain

$$\frac{\left(a\dfrac{d}{dy} - b\dfrac{d}{dx}\right)^\eta}{\{\theta(q)\}^{\frac{\eta}{n(n-1)}}}\,q = \frac{\left(a'\dfrac{d}{dy'} - b'\dfrac{d}{dx'}\right)^\eta}{\{\theta(r)\}^{\frac{\eta}{n(n-1)}}}\,r \ldots\ldots (21).$$

By giving to η the values $1, 2, 3 \ldots n$, we shall obtain a series of homogeneous equations, of which the last but one will, with

(16), determine the linear system, and of which the last will give a relation among the constants. The remaining relations may be deduced, in various ways, from the remaining equations of the above system.

Ex. 1. Suppose the primitive equations to be

$$ax + by = a'x' + b'y' \;\ldots\ldots\ldots\ldots\ldots\; (22),$$

$$Ax^2 + 2Bxy + Cy^2 = A'x'^2 + 2B'x'y' + C'y'^2 \;\ldots\ldots\; (23).$$

Here $q = Ax^2 + 2Bxy + Cy^2$, $\theta(q) = AC - B^2$, $n = 2$, &c.; substituting in (21) and making $\eta = 1$, we have

$$\frac{a(Bx + Cy) - b(Ax + By)}{\sqrt{(AC - B^2)}} = \frac{a'(B'x' + C'y') - b'(A'x' + B'y')}{\sqrt{(AC - B^2)}},$$

or $$\frac{(aB - bA)x + (aC - bB)y}{\sqrt{(AC - B^2)}} = \frac{(a'B' - b'A')x' + (a'C' - b'B')y'}{\sqrt{(AC - B^2)}} \;\ldots (24);$$

this completes the linear system. Again, making $\eta = 2$, we find

$$\frac{a^2C - 2abB + b^2A}{AC - B^2} = \frac{a'^2C' - 2a'b'B' + c'^2A'}{A'C' - B'^2} \;\ldots\ldots\; (25),$$

which expresses the relation among the constants, and may be easily verified by the former method.

We may here observe, that the number of the relations among the constants of a proposed system of homogeneous equations, will be equal to the excess of the number of constants in the functions to be transformed, supposed to be expressed under their most general types, over the square of the number of variables in those functions. This is evident on the common principles of elimination, for the constants in the linear theorems (10), are in number equal to the square of the number of the variables, and it is by an implied elimination of these, that we arrive at the constant relations sought. This rule fails when there are not sufficient data to render the linear system determinate, as in the next example but one; whether in any other case, I have not determined.

Ex. 2. Let the primitive equations be

$$y = mx' + ny' \;\ldots\ldots\; (26),$$

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = a'x'^3 + d'y'^3 \;\ldots\ldots\; (27).$$

Here $\theta(q) = (ad - bc)^2 - 4(b^2 - ac)(c^2 - bd)$, $\theta(r) = a'^2d'^2$, and by (21), the signs being changed for convenience,

$$\frac{\left(\dfrac{d}{dx}\right)^\eta}{\{\theta(q)\}^{\frac{\eta}{6}}}\, q = \frac{\left(n\dfrac{d}{dx'} - m\dfrac{d}{dy'}\right)^\eta}{\{\theta(r)\}^{\frac{\eta}{6}}}\, r.$$

Giving to η the values 1, 2, 3, and performing the differentiations, we have

$$\frac{ax^2+2bxy+cy^2}{\{\theta(q)\}^{\frac{1}{6}}}=\frac{na'x'^2-md'y'^2}{(a'd')^{\frac{1}{3}}}\dots\dots\dots\dots (28),$$

$$\frac{ax+by}{\{\theta(q)\}^{\frac{1}{3}}}=\frac{n^2a'x'+m^2d'y'}{(a'd')^{\frac{2}{3}}}\dots\dots\dots\dots (29),$$

$$\frac{a}{\{\theta(q)\}^{\frac{1}{2}}}=\frac{n^3a'-m^3d'}{a'd'}=\frac{n^3}{d'}-\frac{m^3}{a'}\dots\dots (30).$$

From (28) and (29), by (25) we have

$$\frac{b^2-ac}{\{\theta(q)\}^{\frac{2}{3}}}=\frac{mn}{(a'd')^{\frac{1}{3}}}\dots\dots\dots\dots\dots\dots (31);$$

of the above results, (30) and (31) determine the relations among the constants, and (29) completes the linear system.

We will now examine the forms of solution developed by the principle of our first method; for this purpose cubing (26) our equations become

$$y^3=m^3x'^3+3m^2nx'^2y'+3mn^2x'y'^2+n^3y'^3,$$
$$ax^3-3bx^2y+3cxy^2+dy^3=a'x'^3+3b'x'^2y'+3c'x'y'^2+d'y'^3,$$

b' and c' being supposed to vanish *after the differentiations*, and the symbolical formula for binary systems gives

$$\frac{\left(\frac{d}{d.d}\right)^\eta\theta(q)}{\theta(q)}=\frac{\left(m^3\frac{d}{da'}+m^2n\frac{d}{db'}+mn^2\frac{d}{dc'}+n^3\frac{d}{dd'}\right)^\eta\theta(r)}{\theta(r)}.$$

The first member I shall not develope; in the second we have

$\theta(r)=(a'd'-b'c')^2-\&c.=a'^2d'^2-6a'b'c'd'-3b'^2c'^2+4b'^3d'+4c'^3a'$, giving to η the values 1 and 2, and putting for brevity

$$\theta(q)=\theta,\ \frac{d\theta(q)}{dd}=\theta',\ \frac{d^2\theta(q)}{dd^2}=\theta'',\text{ we obtain}$$

$$\frac{\theta'}{\theta}=\frac{2m^3a'd'^2+2n^3a'^2d'}{a'^2d'^2},$$

$$\frac{\theta''}{\theta}=\frac{2m^6d'^2+8m^3n^3a'd'+2n^6a'^2-12m^3n^3a'd'}{a'^2d'^2},$$

whence, by reduction,

$$\left.\begin{aligned}\frac{m^3}{a'}+\frac{n^3}{d'}&=\frac{\theta'}{2\theta}\\ \frac{m^3}{a'}-\frac{n^3}{d'}&=\sqrt{\frac{\theta''}{2\theta}}\end{aligned}\right\}\dots\dots (32).$$

If $\eta > 2$, the theorem gives $0 = 0$, whence there are no other relations than the above. That they are equivalent to (30) and (31), I have confirmed by actual examination.

The above is a very instructive example. The intelligent reader will observe, that while the method employed in the former of the two solutions, possesses in every other respect the advantage, it is in this particular deficient, that it does not sufficiently limit the number of the final relations; for by employing (29) in the room of (26), or by various other strictly legitimate artifices, we might extend to infinity the series of the constant relations, of which it may however be demonstrated, that two only are independent. This peculiarity will again fall under our notice.

6. As examples of equations with three variables, let us take

Ex. 1. $$ax + by + cz = a'x' + b'y' + c'z' \quad \text{. . (33)},$$
$$Ax^2+By^2+Cz^2+2Dyz+2Exz+2Fxy = A'x'^2 + \&c. \quad \text{. . . . . . (34)}.$$

Here the second method is inapplicable, by the first we get
$$\frac{a^2L + b^2M + c^2N + bcS + acT + abU}{ABC+2DEF-(AD^2+BE^2+CF^2)} = \frac{a'^2L' + \&c.}{A'B'C'+\&c.} \quad \text{. . . . . (35)},$$
L, M, N, &c. as in (57), Part I.

Ex. 2. $$\left.\begin{aligned} ax + by + cz &= a'x' + b'y' + c'z' \\ a_1x + b_1y + c_1z &= a_1'x' + b_1'y' + c_1'z' \\ x^2 + y^2 + z^2 &= x'^2 + y'^2 + z'^2 \end{aligned}\right\} \quad \text{. . . . . (36)};$$

uniting the two methods, we obtain for the constant relations,
$$\left.\begin{aligned} a^2 + b^2 + c^2 &= a'^2 + b'^2 + c'^2 \\ a_1^2 + b_1^2 + c_1^2 &= a_1'^2 + b_1'^2 + c_1'^2 \\ aa_1 + bb_1 + cc_1 &= a'a_1' + b'b_1' + c'c_1' \end{aligned}\right\} \quad \text{. . . . . (37)},$$
and to complete the linear system,
$$(bc_1 - b_1c)x + (ca_1 - c_1a)y + (ab_1 - a_1b)z = (b'c_1' - b_1'c')x' + \&c. \quad \text{. .(38)}.$$

The above, which is a very simple case, is merely given to shew the wide range of the method.

7. Let us next take the general binary system,
$$\Sigma\, ka_1x_1^{\alpha}x_2^{\beta}\ldots x_m^{\mu} = \Sigma\, kb_1y_1^{\alpha}y_2^{\beta}\ldots y_m^{\mu} \text{ for } q=r. \quad \text{. . . (39)},$$
$$Q = R \quad \text{. . . . . (40)},$$
the former equation being of the n^{th}, the latter of the m^{th} degree, supposing $m > n$ and $n > 1$.

The general formula of reduction will be found to be

$$\frac{\left\{\Sigma\left(\frac{d\,\theta(q)}{da_1}\,\frac{d^n}{dx_1{}^\alpha dx_2{}^\beta \ldots dx^\mu{}_m}\right)\right\}^\eta}{\{\theta(q)\}^\eta}\,Q$$

$$= \frac{\left\{\Sigma\left(\frac{d\,\theta(r)}{db_1}\,\frac{d^n}{dy_1{}^\alpha dy_2{}^\beta \ldots dy^\mu{}_m}\right)\right\}^\eta}{\{\theta(r)\}^\eta}\,R \ldots (41);$$

but except when the equations are of a very elevated degree, it will perhaps be more simple to employ directly the general principles of § 4.

Ex. 1. Given the system,

$$ax^2 + 2bxy + cy^2 = a'x'^2 + 2b'x'y' + c'y'^2 \ldots\ldots\ldots (42),$$

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 = A'x'^3 + 3B'x'^2y' + 3C'x'y'^2 + D'y'^3 \ldots (43).$$

taking the second differentials, we have

$$adx^2 + 2bdxdy + cdy^2 = a'dx'^2 + 2b'dx'dy' + c'dy'^2,$$

$$(Ax+By)dx^2 + 2(Bx+Cy)dxdy + (Cx+Dy)dy^2 = (A'x'+B'y')dx'^2 + \&c.$$

and treating these as homogeneous equations of the second degree relatively to dx, dy, dx', dy', we obtain

$$\frac{(Ax+By)(Cx+Dy)-(Bx+Cy)^2}{ac-b^2}$$

$$= \frac{(A'x'+B'y')(C'x'+D'y')-(B'x'+C'y')^2}{a'c'-b'^2}$$

$$\frac{a(Cx+Dy)-2b(Bx+Cy)+c(Ax+By)}{ac-b^2} = \frac{a'(C'x'+D'y')-\&c.}{a'c'-b'^2},$$

or arranging with reference to x and y, x' and y',

$$\frac{(AC-B^2)x^2-2(AD-BC)xy+(BD-C^2)y^2}{ac-b^2} = \frac{(A'C'-B'^2)x'^2-\&c.}{a'c'-b'^2}$$

$$\frac{(aC-2bB+cA)x+(aD-2bC+cB)y}{ac-b^2} = \frac{(a'C'-2b'B'+c'A')x'+\&c.}{a'c'-b'^2}$$

Let $AC-B^2=p$. $AD-BC=q$. $BD-C^2=r$.
$aC-2bB+cA=s$. $aD-2bC+cB=t$.
$A'C'-B'^2=p'$. &c. &c.

then have we the following system of equations,

$$ax^2 + 2bxy + cy^2 = a'x'^2 + 2b'x'y' + c'y'^2 \ldots\ldots (44),$$

$$\frac{px^2-qxy+ry^2}{ac-b^2} = \frac{p'x'^2-q'x'y'+r'y'^2}{a'c'-b'^2} \ldots\ldots (45).$$

$$\frac{sx+ty}{ac-b^2}=\frac{s'x'+t'y'}{a'c'-b'^2}\ \ldots\ldots\ (46).$$

From (44) and (45), which are homogeneous of the second degree,

$$\frac{q^2-4pr}{(ac-b^2)^3}=\frac{q'^2-4p'r'}{(a'c'-b'^2)^3}\ \ldots\ldots\ (47),$$

$$\frac{ar-bq+cp}{(ac-b^2)^2}=\frac{a'r'-b'q'+c'p'}{(a'c'-b'^2)^2}\ \ldots\ldots\ (48).$$

Also, from (44) and (46), *vide* Ex. 1, §. 5,

$$\frac{at^2-2bst+cs^2}{(ac-b^2)^3}=\frac{a'b'^2-2b's't'+c's'^2}{(a'c'-b'^2)^3}\ \ldots\ldots\ (49),$$

$$\frac{(at-bs)\,x+(bt-cs)\,y}{(ac-b^2)^{\frac{3}{2}}}=\frac{(a't'-b's')\,x'+(b't'-c's')y'}{(a'c'-b'^2)^{\frac{3}{2}}}\ \ldots\ldots\ (50).$$

Of the above results (46) and (50) determine the linear system, (47) (48) and (49) express the relations among the constants. Other forms of solution may be found in infinite variety; thus a relation may be found from (45) and (46), but they will all be combinations of those we have already obtained. Meanwhile there is nothing in the process which appears to indicate when it is necessary to stop, and what is the nature of that functional connexion which must exist among the interminable series of equations, to which, if continued, it would give birth. To the discussion of this question, I would especially direct the attention of those who may be disposed to take up the subject.

8. Linear transformations have hitherto been chiefly applied to the purpose of taking away from a proposed homogeneous function, those terms which involve the products of the variables. It may be observed that this problem resolves itself into two principal cases: the first is that in which the transformations, besides being linear, are understood to represent a geometrical change of axes, or are such as to involve an obvious extension of this analogy; the second case is when no other condition than that of linearity is introduced. It is to the former of the above cases, and to that only as developed in the first of the subjoined examples, that the efforts of analysts appear to have been hitherto directed.

Ex. 1. To transform the homogeneous function Q, of the second degree, with m variables, $x_1, x_2, \ldots\ldots x_m$, to the form $B_1y_1^2+B_2y_2^2\ldots\ldots+B_my_m^2$, conformably with the condition

$$x_1^2+x_2^2\ldots\ldots+x_m^2=y_1^2+y_2^2\ldots\ldots+y_m^2.$$

Representing the above condition by $q = r$, form the equation $\theta(Q + hq) = 0$; the values of h, taken negatively, will determine $B_1, B_2, \ldots . B_m$.

Ex. 2. To exhibit, under a general theorem, all the linear systems by which the function $ax^2 + 2bxy + cy^2$, may be reduced to the form $a'x'^2 + c'y'^2$, a' and c' being given.

Let m and n be any two quantities satisfying the condition

$$\frac{m^2}{a'} + \frac{n^2}{c'} = \frac{a}{ac - b^2} \ldots . (51);$$

then are the general linear forms in question

$$\left.\begin{aligned} y &= mx' + ny' \\ \frac{ax + by}{\sqrt{(ac - b^2)}} &= \frac{na'x' - mc'y'}{\sqrt{(a'c')}} \end{aligned}\right\} \ldots \ldots (52);$$

these results are deduced from Ex. 1, § 5.

Ex. 3. To take away the products of the variables from a proposed homogeneous function, of the form

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

by transformations equivalent to a geometrical change of axes.

Let θ represent the inclination of the axes x and y, θ' that of the unknown axes x' and y', and $a'x'^3 + d'y'^3$ the transformed function; then, by the aid of ten subsidiary quantities $p, q, r, s, t, P, Q, R, S, T$, the solution will be conveniently expressed in the annexed forms,

$$\left.\begin{aligned} &p = ac - b^2,\ q = ad - bc,\ r = bd - c^2, \\ &s = a - 2b \cos \theta + c,\ t = b - 2c \cos \theta + d, \\ &P = \sqrt{(q^2 - 4pr)},\ Q = p - q \cos \theta + r,\ R = s^2 - 2st \cos \theta + t^2. \\ &\sin \theta' = \frac{P \sin \theta}{\sqrt{\{P^2 (\sin \theta)^2 + Q^2\}}}, \\ &S = R\left(\frac{\sin \theta'}{\sin \theta}\right)^6 - 2Q\left(\frac{\sin \theta'}{\sin \theta}\right)^4 + 2P\left(\frac{\sin \theta'}{\sin \theta}\right)^3, \\ &T = R\left(\frac{\sin \theta'}{\sin \theta}\right)^6 - 2Q\left(\frac{\sin \theta'}{\sin \theta}\right)^4 - 2P\left(\frac{\sin \theta'}{\sin \theta}\right)^3, \\ &a' = \frac{\sqrt{S} + \sqrt{T}}{2},\ d' = \frac{\sqrt{S} - \sqrt{T}}{2}, \end{aligned}\right\} \ldots (53);$$

and for the linear relations,

$$\left.\begin{aligned} \frac{sx + ty}{(\sin \theta)^2} &= \frac{a'x' + d'y'}{(\sin \theta')^2}, \\ \frac{(t - s\cos \theta)x - (s - t\cos \theta)y}{(\sin \theta)^3} &= \frac{(d' - a'\cos \theta')x' - (a' - d'\cos \theta')y'}{(\sin \theta')^3} \end{aligned}\right\} \ldots (54)$$

When the original axes are rectangular, the above results are greatly simplified. The reader will perceive that they are deduced from Ex. 1, § 7.

We take as a numerical illustration the equation

$$37x^3 - 36x^2y - 36xy^2 + 37y^3 = 1,$$

the axes x and y being rectangular.

Here $a = 37$, $b = -12$, $c = -12$, $d = 37$. The resulting form is found to be

$$(\tfrac{7}{5})^3 x'^3 + (\tfrac{7}{5})^3 y'^3 = 1,$$

with the linear relations

$$x = \tfrac{4}{5} x' + \tfrac{3}{5} y';$$
$$y = \tfrac{3}{5} x' + \tfrac{4}{5} y'.$$

Ex. 4. To transform the function, $ax^3 + 3bx^2y + 3cxy^2 + dy^3$, to the form $a'x'^3 + d'y'^3$, a' and d' being given, and the transformation unrestricted by any other condition than that of linearity.

The direct solution of this problem is contained in the formulæ of Ex. 2, § 5, which shew that if m and n be so determined as to satisfy the conditions,

$$\frac{m^3}{a'} + \frac{n^3}{d'} = \frac{\theta'}{2\theta}, \quad \frac{m^3}{a'} - \frac{n^3}{d'} = \sqrt{\frac{\theta''}{2\theta}} \;\ldots\ldots\; (55);$$

then the linear system in question will be

$$\left.\begin{aligned} y &= mx' + ny' \\ \frac{ax + by}{(\theta)^{\frac{1}{3}}} &= \frac{n^2a'x' + m^2d'y'}{(a'd')^{\frac{2}{3}}} \end{aligned}\right\} \;\ldots\ldots\; (56).$$

9. The doctrine of linear transformations may be elegantly applied to the solution of algebraic equations. The following example, in which I shall apply the above theorems, will clearly shew the nature of the connexion.

Ex. The most general form of the cubic equation is

$$av^3 + 3bv^2 + 3cv + d = 0 \;\ldots\; (57);$$

the simplest of possible forms is

$$v'^3 - 1 = 0 \;\ldots\ldots\; (58),$$

giving $v' = 1$. If by linear transformation we can reduce (57) to (58), the solution of the former will be derived from that of the latter. To effect this we first render them homogeneous by putting $v = \dfrac{x}{y}$, $v' = \dfrac{x'}{y'}$. The problem is then reduced to the discussion of the equation

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = x'^3 - y'^3 \;\ldots\ldots\; (59).$$

Here, since $a' = 1$, $d' = -1$, we have, by the theorems above,

$$m^3 - n^3 = \frac{\theta'}{2\theta};\ m^3 + n^3 = \sqrt{\left(\frac{\theta''}{2\theta}\right)} \ldots\ldots (60),$$

$$y = mx' + ny' \ldots\ldots (61),$$

$$\frac{ax + by}{\sqrt[3]{\theta}} = n^2x' - m^2y' \ldots\ldots (62).$$

From (60) we find

$$m = \sqrt[3]{\left\{\frac{\theta' + \sqrt{(2\theta\theta'')}}{4\theta}\right\}}\quad n = \sqrt{\left\{\frac{-\theta' + \sqrt{(2\theta\theta'')}}{4\theta}\right\}}.$$

Dividing (62) by (61), and making $\dfrac{x}{y} = v$, $\dfrac{x'}{y'} = v' = 1$, we find

$$\frac{av + b}{\sqrt[3]{\theta}} = \frac{n^2 - m^2}{n + m} = n - m;$$

$$\therefore\ av + b = \sqrt{\left\{\frac{-\theta' + \sqrt{(2\theta\theta'')}}{4}\right\}} + \sqrt[3]{\left\{\frac{-\theta' - \sqrt{(2\theta\theta'')}}{4}\right\}} \ldots (63),$$

in which it is only necessary to observe, that

$$\theta = \theta(q) = (ad - bc)^2 - 4(b^2 - ac)(c^2 - bd),$$

$$\theta' = \frac{d\,\theta(q)}{dd} = 2(a^2d - 3abc + 2b^3),\qquad \theta'' = \frac{d^2\,\theta(q)}{dd^2} = 2a^2.$$

To extend this investigation to the equations of the fourth and fifth degree, will require the previous determination of $\theta(q)$ for those cases, a question tedious but not difficult, and to which either the method described in Part I., or the ingenious modes of elimination devised by Professor Sylvester, may be applied. As this question is of fundamental importance, and needs to be determined but once, it is much to be desired that some one, possessed of leisure, would undertake its discussion.

An equally important subject of inquiry presents itself in the connexion between linear transformations and an extensive class of theorems depending on partial differentials, particularly such as are met with in Analytical Geometry. It is not my intention to enter into the subject in this place, nor have I leisure either to pursue the inquiry, or to elucidate my present views in a separate paper. To those who may be disposed to engage in the investigation, it will, I believe, present an ample field of research and discovery. It is almost needless to observe, that any additional light which may be thrown on the general theory, and especially as respects the properties of the function $\theta(q)$, will tend to facilitate our further progress, and to extend the range of useful applications.

Lincoln, October 21st, 1841.

IV.—A METHOD OF OBTAINING ANY ROOT OF A NUMBER IN THE FORM OF A CONTINUED FRACTION.*

THE principle of the following method of approximating to any root of a number, will be best exhibited by taking first the simplest case, that of the square root.

Let N be the number, and let $N = a^2 + b$, a^2 being the square number next less than N. Then identically, $\sqrt{(N)} = a + \sqrt{(a^2+b)} - a$. Now $\sqrt{(a^2+b)} - a$ is less than unity, and may therefore be assumed equal to the continued fraction

$$\frac{1}{p+}\,\frac{1}{q+}\,\frac{1}{r+\&c.}\,.$$

Hence $\sqrt{(a^2+b)} - a$ is less than $\frac{1}{p}$ and greater than $\frac{1}{p+1}$, so that p is the greatest whole number that satisfies the inequality

$$\sqrt{(a^2+b)} - a < \frac{1}{p}$$

By squaring, $a^2 + b < a^2 + \frac{2a}{p} + \frac{1}{p^2}$; or $bp^2 - 2ap < 1$; whence p is readily found by trial. Again, $\sqrt{(a^2+b)} - a$ is greater than $\cfrac{1}{p+\cfrac{1}{q}}$ and less than $\cfrac{1}{p+\cfrac{1}{q+1}}$. Hence q is the greatest whole number that satisfies the inequality $\sqrt{(a^2+b)} - a > \cfrac{1}{p+\cfrac{1}{q}}$; or $b\left(p+\frac{1}{q}\right)^2 - 2a\left(p+\frac{1}{q}\right) > 1$. By means of this inequality, q may be found by trial when p is known. It is evident that the successive inequalities may be formed by substituting $p + \frac{1}{q}$ for p in the first, $q + \frac{1}{r}$ for q in the second, and so on, and changing the sign $>$ or $<$ at each substitution. This consideration will facilitate the numerical calculation, as will be seen by an example.

Let $N = 11$. Then $a = 3$, $b = 2$, and the first inequality is $2p^2 - 6p < 1$. Hence $p = 3$. The second inequality is therefore

* From a Correspondent.

$2\left(3+\frac{1}{q}\right)^2 - 6\left(3+\frac{1}{q}\right) > 1$, which gives $q^2 - 6q < 2$; hence $q = 6$. The third inequality is $\left(6+\frac{1}{r}\right)^2 - 6\left(6+\frac{1}{r}\right) > 2$, which gives $2r^2 - 6r < 1$. As this is the same as the first, the operations will recur, and we therefore have

$$\sqrt{(11)} = 3 + \frac{1}{3+}\ \frac{1}{6+}\ \frac{1}{3+\&c.}.$$

The way of proceeding in approximating to the cube root of a number is precisely analogous to that above. Let $N = a^3 + b$; then, as before, $\sqrt[3]{(a^3+b)} - a < \frac{1}{p}$. Hence $a^3 + b < a^3 + \frac{3a^2}{p} + \frac{3a}{p^2} + \frac{1}{p^3}$; or, $bp^3 - 3a^2p^2 - 3ap < 1$; and p is the greatest whole number that satisfies this inequality. Let, for example, $N = 10$; then $a = 2$, $b = 2$, and $2p^3 - 12p^2 - 6p < 1$. Hence $p = 6$. By substituting $6 + \frac{1}{q}$ for p in the above inequality, and changing $<$ into $>$, it will be found that $37q^3 - 66q^2 - 24q < 2$. Hence $q = 2$. The next inequality is $18r^3 - 156r^2 - 156r < 37$, from which $r = 9$, and so on. Hence approximately

$$\sqrt[3]{(10)} = 2 + \frac{1}{6+}\ \frac{1}{2+}\ \frac{1}{9} = \frac{265}{123};$$

this result is true to four places of decimals.

The same method applied in extracting the fourth root of 20, gives the approximate value

$$2 + \frac{1}{8+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{1} = \frac{129}{61},$$

which is very nearly true to five places of decimals; the next quotient is found to be 22.

It is evident that this process may be employed to approximate to any root of a fraction. Let, for instance,

$$\sqrt{\left(\frac{30}{7}\right)} = 2 + \frac{1}{p+}\ \frac{1}{q+\&c.};\ \text{then}\ \sqrt{\left(\frac{30}{7}\right)} - 2 < \frac{1}{p}.$$

Whence $2p^2 - 28p < 7$, and $p = 14$. The second inequality is therefore $2\left(14+\frac{1}{q}\right)^2 - 28\left(14+\frac{1}{q}\right) > 7$, which gives $7q^2 - 28q < 2$. Hence $q = 4$. The third is, $2r^2 - 28r < 7$, and is the same as the first. Consequently

$$\sqrt{\left(\frac{30}{7}\right)} = 2 + \frac{1}{14+}\ \frac{1}{2+}\ \frac{1}{14+}\ \frac{1}{2+\&c.},$$

a result easily verified by obtaining in the usual manner the value of this recurring continued fraction.

Let it be required to approximate to $\sqrt[3]{(\frac{2}{3})}$ in a continued fraction. As this quantity is less than unity we may assume

$$\sqrt[3]{\left(\frac{2}{3}\right)} = \frac{1}{p+}\;\frac{1}{q+}\;\frac{1}{r+\&c.}\,.$$

Hence p is the greatest whole number that satisfies the condition $\sqrt[3]{\left(\frac{2}{3}\right)} < \frac{1}{p}$. The successive inequalities and the resulting values of p, q, r, &c. will be found to be as follows:—

$$\begin{array}{lcll} 2p^3 & < & 3, & \therefore p = 1 \\ q^3 - 6q^2 - 6q & < & 2, & \therefore q = 6 \\ 38r^3 - 30r^2 - 12r & < & 1, & \therefore r = 1 \\ 5s^3 - 42s^2 - 84s & < & 38, & \therefore s = 10. \end{array}$$

Hence $\sqrt[3]{\left(\frac{2}{3}\right)} = \frac{1}{1+}\;\frac{1}{6+}\;\frac{1}{1+}\;\frac{1}{10+} = \frac{76}{87}$ nearly; this value is correct to four places of decimals.

It appears therefore that the method here proposed gives the means of throwing any root of either an integral or a fractional quantity into the form of a continued fraction, and consequently of approximating to the root with any required degree of accuracy.

C.

V.—NOTES ON SOME POINTS IN FORMAL OPTICS.*

1. *Construction for the place of the primary focal line after oblique reflexion at a spherical surface.*

Let E be the centre of the surface, P the focus, PR the axis of an incident pencil meeting the surface in R, RQ the axis of the reflected pencil. From E draw EF perpendicular to PR meeting PR in F; from F draw FG perpendicular to ER meeting ER in G, and draw the straight line PG meeting RQ in Q. Then Q will be the place of the primary focal line. For by the construction Q would be the place of the secondary focal line, after reflexion at R of a pencil having P for its focus, from a surface having G for its centre. Therefore, if

$$ER = r,\; PRE = \phi,\; PR = u,\; QR = v,\; \frac{1}{v} + \frac{1}{u} = \frac{2}{RG}\cos\phi.$$

But $RG = r(\cos\phi)^2$, $\therefore \frac{1}{v} + \frac{1}{u} = \frac{2}{r\cos\phi}$, $\therefore Q$ is the place of the primary focal line.

* From a Correspondent.

2. *Construction for the place of the primary focal line after oblique refraction at a spherical surface.*

Let E be the centre of the surface, P the focus, PR the axis of an incident pencil meeting the surface in R, RQ the axis of the refracted pencil. From P draw PF perpendicular to PR meeting RE in F; from F draw FG perpendicular to RE meeting PR in G; draw the straight line EG meeting RQ in K; from K draw KH perpendicular to RE meeting RE in H; from H draw HQ perpendicular to RQ meeting RQ in Q. Then Q will be the place of the primary focal line. For by the construction Q would be the place of the secondary focal line of a pencil having G for its focus, and GR for its axis after refraction at R. Therefore, if $RE = r$, $PRE = \phi$, $QRE = \phi'$, $PR = u$, $QR = v$, $\sin\phi = \mu \sin\phi'$,

$$\frac{\mu}{KR} - \frac{1}{GR} = \frac{1}{r}(\mu\cos\phi' - \cos\phi), u = GR(\cos\phi)^2, v = KR(\cos\phi')^2.$$

Therefore $$\frac{\mu(\cos\phi')^2}{v} - \frac{(\cos\phi)^2}{u} = \frac{1}{r}(\mu\cos\phi' - \cos\phi).$$

Therefore Q is the place of the primary focal line.

3. *Construction for the place of the primary focal line after oblique refraction at a plane surface.*

Let P be the focus of the incident pencil, PR its axis meeting the surface in R; RQ the axis of the refracted pencil. Draw RF perpendicular to the surface; draw PF perpendicular to PR meeting RF in F; draw FG perpendicular to RF meeting RP in G; through G draw GK parallel to RF meeting RQ in K; draw KH perpendicular to RF meeting RF in F, and HQ perpendicular to RQ meeting RQ in Q. Then Q will be the place of the primary focal line. For if $PRF = \phi$, $QRF = \phi'$, $\sin\phi = \mu\sin\phi'$, $PR = u$, $QR = v$, $KR = \mu GR$, $u = GR(\cos\phi)^2$, $v = KR(\cos\phi')^2$.

Therefore $$\mu\frac{(\cos\phi')^2}{v} - \frac{(\cos\phi)^2}{u} = 0.$$

Therefore Q is the place of the primary focal line.

W. H. M.

VI.—ON ELLIPTIC FUNCTIONS.

By B. Bronwin.

Sir James Ivory has greatly simplified the Theory of Elliptic Functions as given by M. Jacobi. He has also applied the theory to the case where the index of multiplica-

tion is an even number. But the case of which M. Jacobi has treated is susceptible of still further simplification.

The following auxiliary formulæ are from Jacobi, page 32.

Put $am\,.\,u = a$, $am\,.\,v = b$, $am\,.\,(u+v) = \sigma$, $am\,.\,(u-v) = \theta$; then if $\Delta a = \sqrt{(1 - k^2\sin^2 a)}$, $\Delta b = \sqrt{(1-k^2\sin^2 b)}$, $k^2 + k'^2 = 1$,

$$\left.\begin{aligned}
\sin\sigma &= \frac{\sin a\cos b\Delta b + \sin b\cos a\,\Delta a}{1 - k^2\sin^2 a\sin^2 b}\\
\cos\sigma &= \frac{\cos a\cos b - \sin a\sin b\,\Delta a\Delta b}{1 - k^2\sin^2 a\sin^2 b}\\
\Delta\sigma &= \frac{\Delta a\Delta b - k^2\sin a\sin b\cos a\cos b}{1 - k^2\sin^2 a\sin^2 b}\\
\sin\theta &= \frac{\sin a\cos b\Delta b - \sin b\cos a\Delta a}{1 - k^2\sin^2 a\sin^2 b}\\
\cos\theta &= \frac{\cos a\cos b + \sin a\sin b\Delta a\Delta b}{1 - k^2\sin^2 a\sin^2 b}\\
\Delta\theta &= \frac{\Delta a\Delta b + k^2\sin a\sin b\cos a\cos b}{1 - k^2\sin^2 a\sin^2 b}\\
\sin\sigma\sin\theta &= \frac{\sin^2 a - \sin^2 b}{1 - k^2\sin^2 a\sin^2 b}\\
\cos\sigma\cos\theta &= \frac{\cos^2 a\Delta^2 b - k'^2\sin^2 b}{1 - k^2\sin^2 a\sin^2 b}\\
\Delta\sigma\Delta\theta &= \frac{\Delta^2 a\cos^2 b + k'^2\sin^2 b}{1 - k^2\sin^2 a\sin^2 b}
\end{aligned}\right\}\ \ldots\ (\text{A}).$$

The transformation to be effected is

$$\frac{d\psi}{\sqrt{(1 - h^2\sin^2\psi)}} = \frac{\beta d\phi}{\sqrt{(1 - k^2\sin^2\phi)}},\ \text{or}\ dv = \beta du\ \ldots\ (\text{B}).$$

Here $\psi = am\,.\,v$, $\phi = am\,.\,u$. Let $v = H$ when $\psi = \frac{\pi}{2}$, $u = K$ when $\phi = \frac{\pi}{2}$, $h^2 + h'^2 = 1$; and let H' and K' be what H and K become when h is changed into h' and k into k'. Make $\omega = \frac{K}{n}$, n being an odd integer. Moreover when $u = 0$, ω, 2ω, 3ω, &c; suppose $v = 0$, H, $2H$, $3H$, &c. And to abridge as much as possible we shall put $s\,.\,a$ for sin amplitude, $c\,.\,a$ for cos *am*, and $t\,.\,a$ for tan *am*.

Assume

$$s.a.v = \frac{s.a.u\ s.a\,(u+2\omega)\,s.a\,(u+4\omega)\ldots s.a\left\{u + 2\,(n-1)\,\omega\right\}}{s.a.\omega\ s.a.3\omega\ s.a.5\omega\ldots s.a\,(2n-1)\,\omega}\ \ldots(1),$$

$$c.a.v = \frac{c.a.u\, c.a\,(u+2\omega)\, c.a(u+4\omega) \ldots c.a\, \{u+2\,(n-1)\,\omega\}}{c.a.2\omega\; c.a.4\omega\; c.a.6\omega \ldots c.a.2\,(n-1)\,\omega} \ldots (2).$$

When u is changed into $u + 2\omega$, each factor of (1) and (2) goes into the succeeding one, and the last into the first with a contrary sign. The second members therefore remain unchanged except as to the sign. Consequently if $u = 0,\ 2\omega,\ 4\omega$, &c.; $\sin am\,.\,v = 0$; $\cos am\,.\,v = \pm 1$. If $u = \omega,\ 3\omega,\ 5\omega$, &c.; $\sin am\,.\,v = \pm 1$; $\cos am\,.\,v = 0$, because some factor in the numerator of (2) becomes $\cos am\,.\,n\omega = \cos am\,.\,K = 0$. The second members of (1) and (2) therefore are suitable expressions of the values of the first.

In (1) change u into $u + K$, v into $v + nH$; then by the first of (A) we have

$$\sin am\,.(v+nH) = \pm\, \frac{\cos am\,.\,v}{\Delta am\,.\,v},\ \sin am\,.(u+K) = \frac{\cos am\,.\,u}{\Delta am\,.\,u}.$$

The other factors will be similarly changed; and (1) becomes, putting A for its denominator,

$$\pm\, \frac{c\,.\,a\,.\,v}{\Delta\,.\,a\,.\,v} = \frac{c\,.\,a\,.\,uc\,.\,a\,(u+2\omega) \ldots c\,.\,a\,\{u+2\,(n-1)\,\omega\}}{A\Delta\,.\,a\,.\,u\Delta\,.\,a(u+2\omega) \ldots \Delta\,.\,a\,\{u+2\,(n-1)\,\omega\}}.$$

Multiplying this, member by member, by (1); we have

$$\pm\, A^2\, \frac{s.a.v\; c.a.v}{\Delta.a.v} = \frac{s.a.u\; c.a.u}{\Delta.a.u}\,.\,\frac{s.a\,(u+2\omega)\; c.a\,(u+2\omega)}{\Delta.a\,(u+2\omega)} \ldots (3).$$

This result is derived entirely from (1), and must therefore give the same relation between v and u which (1) does.

In (2) change u into $u + K$, v into $v + nH$. Then by the second of (A) we find

$$\cos am(v+nH) = \mp\, \frac{h' \sin am.v}{\Delta am.v},\ \cos am(u+K) = -\,\frac{k' \sin am.u}{\Delta am.u}, \text{\&c.}$$

Thus (2) becomes, putting B for its denominator,

$$\pm\, \frac{h' s.a.v}{\Delta.a.v} = \frac{k'^n s.a.us.a\,(u+2\omega) \ldots\ldots s.a.\,\{u+2\,(n-1)\,\omega\}}{B\Delta.a.u\Delta.a(u+2\omega) \ldots\ldots \Delta.a\,\{\,u+2\,(n-1)\,\omega\}}.$$

Multiplying this by (2), member by member; we obtain

$$\pm\, \frac{B^2 h'}{k'^n}\,.\,\frac{s.a.vc.a.v}{\Delta.a.v} = \frac{s.a.uc.a.u}{\Delta.a.u}\,.\,\frac{s.a\,(u+2\omega)\; c.a\,(u+2\omega)}{\Delta.a\,(u+2\omega)} \ldots (4).$$

This last is derived entirely from (2), and must therefore give the same relation between v and u which (2) does. But if $A^2 = \dfrac{B^2 h'}{k'^n}$, (4) is the same as (3). Therefore (1) and (2) give the same relation between v and u, or they are derivable from each other.

The equation $A^2 k'^n = B^2 h'$ gives

$$h' = k'^n \left\{ \frac{s.a.\omega \; s.a.3\omega \ldots s.a\,(2n-1)\,\omega}{c.a.2\omega \; c.a.4\omega \ldots c.a\,2\,(n-1)\,\omega} \right\}^2$$

$$= k'^n \left\{ \frac{s.a.\omega \; s.a.3\omega \ldots s.a\,(n-2)\,\omega}{c.a.2\omega \; c.a.4\omega \ldots c.a\,(n-1)\,\omega} \right\}^4$$

$$= \frac{k'^n}{\{\Delta\,(2\omega)\,\Delta\,(4\omega) \ldots \Delta\,(n-1)\,\omega\}^4}.$$

If in (B) we make v and u infinitely small, we find

$$\beta = \frac{s.a.2\omega \; s.a.4\omega \ldots s.a.2\,(n-1)\,\omega}{s.a.\omega \; s.a.3\omega \ldots s.a\,(2n-1)\,\omega}$$

$$= \left\{ \frac{s.a.2\omega \; s.a.4\omega \ldots s.a\,(n-1)\,\omega}{s.a.\omega \; s.a.3\omega \ldots s.a\,(n-2)\,\omega} \right\}^2.$$

We notice here a particular property of (1). By (A) we find $s.a\,(u+2\omega)\,s.a\,\{u+2\,(n-1)\,\omega\} = s.a\,(2\omega+u)\,s.a\,(2\omega-u)$ $= \dfrac{s^2.a.2\omega - s^2.a.u}{1 - k^2 s^2.a.2\omega \; s^2.a.u}$. If in this we change $s.a.u$ into $\dfrac{1}{ks.a.u}$, it becomes

$$\frac{1}{k^2} \cdot \frac{1 - k^2 s^2.a.2\omega \; s^2.a.u}{s^2.a.2\omega - s^2.a.u} = \frac{1}{k^2} \cdot \frac{1}{s.a\,(u+2\omega)\; s.a\{u+(2-1)\omega\}}.$$

The same change would take place in every other corresponding pair of factors. Hence it is easily seen, that to change $s.a.u$ into $\dfrac{1}{ks.a.u}$, we should change $s.a.v$ into $\dfrac{1}{hs.a.v}$, if we make

$$h = k^n \{s.a.\omega \; s.a.3\omega \ldots s.a(2n-1)\,\omega\}^2$$

$$= k^n \{s.a.\omega \; s.a.3\,\omega \ldots s.a\,(n-2)\,\omega\}^4.$$

If in (2), we change $s.a.u$ into $\dfrac{1}{ks.a.u}$, $s.a.v$ into $\dfrac{1}{hs.a.v}$; and compare the result with a similar result before obtained; we shall see that h is the quantity so denominated in (B). From what has been done we easily derive the following:

$$\left.\begin{aligned}
s.a.v &= \sqrt{\left(\frac{k^n}{h}\right)} \cdot s.a.u \; s.a\,(u+2\omega) \ldots s.a\,\{u+2\,(n-1)\,\omega\} \\
c.a.v &= \sqrt{\left(\frac{h'k^n}{hk'^n}\right)} \cdot c.a.u \; c.a\,(u+2\omega) \ldots c.a\,\{u+2(n-1)\,\omega\}. \\
t.a.v &= \sqrt{\left(\frac{k'^n}{h'}\right)} \cdot t.a.u \; t.a\,(u+2\omega) \ldots t.a\,\{u+2\,(n-1)\,\omega\}. \\
\Delta.a.v &= \sqrt{\left(\frac{h'}{k'^n}\right)} \cdot \Delta.a.u \; \Delta.a\,(u+2\omega) \ldots \Delta.a\,\{u+2\,(n-1)\,\omega\}.
\end{aligned}\right\} (C).$$

If we make $x = s\,.\,a\,.\,u$; the first of (C), when developed, becomes $\sqrt{\left(\frac{h}{k}\right)}\,.\,s\,.\,a\,.\,v = xP\left(\frac{s^2\,.\,a\,.\,2r\omega - x^2}{1 - k^2x^2s^2\,.\,a\,.\,2r\omega}\right)$;

or $xP(x^2 - s^2\,.\,a\,.\,2r\omega) - \frac{h\beta}{k}s\,.\,a\,.\,vP\left(x^2 - \frac{1}{k^2s^2\,.\,a\,.\,2r\omega}\right) = 0$, where P denotes the continued product, giving to r all the integer values from 1 to $\frac{n-1}{2}$ inclusive. The roots or values of x in this equation are

$$s\,.\,a\,.\,u,\ s\,.\,a(u+4\omega),\ s\,.\,a(u+8\omega)\ldots s\,.\,a\left\{u+2(n-1)\omega\right\};$$
$$\text{and}\ -s\,.\,a(u+2\omega),\ -s\,.\,a(u+6\omega)\ldots -s\,.\,a\left\{u+2(n-2)\omega\right\};$$
$$\text{or}\ \ s\,.\,a(u-4\omega),\ s\,.\,a(u-8\omega)\ldots s\,.\,a\left\{u-2(n-1)\omega\right\}.$$

The coefficient of the second term of the above equation with its sign changed is equal to the sum of these roots. In like manner by making $x = c\,.\,a\,.\,u$, $x = t\,.\,a\,.\,u$, $x = \Delta\,.\,a\,.\,u$; we shall obtain similar results for these quantities. These results are

$$\left.\begin{aligned}
&\frac{h\beta}{k}\,.\,s.a.v = s.a.u + \Sigma\, s.a(u+4r\omega) + \Sigma\, s.a(u-4r\omega)\\
&\frac{h\beta}{k}\,.\,c.a.v = c.a.u + \Sigma\, c.a(u+4r\omega) + \Sigma\, c.a(u-4r\omega)\\
&\frac{h'\beta}{k'}\,.\,t.a.v = t.a.u + \Sigma\, t.a(u+2r\omega)\\
&\beta\ \Delta.a.v = \Delta.a.u + \Sigma\,\Delta.a(u+2r\omega)
\end{aligned}\right\}\ldots\text{(D)}.$$

In the two first of (D) r is all the numbers $1, 2, \ldots \frac{n-1}{2}$; in the two last it is all the numbers $1, 2 \ldots n-1$.

By making $x = \frac{1}{s.a.u}$, $x = \frac{1}{c.a.u}$, &c. we might find a great many more formulæ similar to (D). But they may be more easily found from (D) by changing u into $u + K$, v into $v + nH$, or changing $s.a.u$ into $\frac{1}{k\,s.a.u}$, $s.a.v$ into $\frac{1}{h\,s.a.v}$; or by both these operations combined. In all the preceding formulæ, I have omitted the ambiguous sign $\pm$ when it occurs, thinking it perplexing and useless.

The first and second of (C) give

$$\sin\psi = \sqrt{\left(\frac{k}{h}\right)}\,.\,\sin\phi P\left(\frac{s^2\,.\,a\,.\,2r\omega - \sin^2\phi}{1 - k^2s^2\,.\,a\,.\,2r\omega\,\sin^2\phi}\right)$$
$$= \frac{\beta\tan\phi}{\sqrt{(1+\tan^2.\,\phi)}}\,.\,P\left(\frac{1-\cot^2 a.2r\omega\tan^2\phi}{1+\Delta^2 a.2r\omega\tan^2\phi}\right),$$

$$\cos\psi = \sqrt{\left(\frac{k^n}{h}\right)} \cdot \cos\phi P\left(\frac{s^2 . a . (2r-1)\omega - \sin^2\phi}{1 - k^2 s^2 . a . 2r\omega \sin^2\phi}\right)$$

$$= \frac{1}{\sqrt{(1+\tan^2\phi)}} . P\left(\frac{1 - \cot^2 a\,(2r-1)\,\omega \tan^2\phi}{1 + \Delta^2 . a . 2r\omega \tan^2\phi}\right),$$

and therefore $\tan\psi = \beta \tan\phi . P\left(\frac{1 - \cot^2 a . 2r\omega \tan^2\phi}{1 - \cot^2 a(2r-1)\omega \tan^2\phi}\right)$.

Let R be the denominator of $\sin\psi$ and $\cos\psi$; then $\sin^2\psi + \cos^2\psi = 1$;

$$R^2 \sin^2\psi + R^2 . \cos^2\psi = R^2 = (1 + \tan^2\phi) . P(1 + \Delta^2 . a . 2r\omega \tan^2\phi)^2.$$

Each factor of the first member of this last must be equal to the product of all those factors of the second member which divide it. Now $R\cos\psi + R\sin\psi\sqrt{(-1)} = 0$ gives $\tan\psi = \sqrt{(-1)}$, $R\cos\psi - R\sin\psi\sqrt{(-1)} = 0$ gives $\tan\psi = -\sqrt{(-1)}$; also $1 + \Delta . a . 2r\omega \tan\phi\sqrt{(-1)} = 0$ gives $\tan\phi = \frac{\sqrt{(-1)}}{\Delta . a . 2r\omega}$, which substituted in the expression of $\tan\psi$ gives it affirmative.

$$1 - \Delta . a . 2r\omega \tan\phi\sqrt{(-1)} = 0 \text{ gives } \tan\phi = \frac{-\sqrt{(-1)}}{\Delta . a . 2r\omega},$$

which gives $\tan\psi$ negative. The factors of the second member therefore which have their second term positive, divide that factor of the first whose second term is positive; and those which have the second term negative, divide that whose second term is negative. Consequently we have

$$R\cos\psi + R\sin\psi\sqrt{(-1)}$$
$$= \{1 + \tan\phi\sqrt{(-1)}\} . P\{1 + \Delta . a . 2r\omega \tan\phi\sqrt{(-1)}\}^2,$$

$$R\cos\psi - R\sin\psi\sqrt{(-1)}$$
$$= \{1 - \tan\phi\sqrt{(-1)}\} . P\{(1 - \Delta . a . 2r\omega \tan\phi\sqrt{(-1)}\}^2,$$

$$\frac{1 + \tan\psi\sqrt{(-1)}}{1 - \tan\psi\sqrt{(-1)}}$$
$$= \frac{1 + \tan\phi\sqrt{(-1)}}{1 - \tan\phi\sqrt{(-1)}} . P\left\{\frac{1 + \Delta . a . 2r\omega \tan\phi\sqrt{(-1)}}{1 - \Delta . a . 2r\omega \tan\phi\sqrt{(-1)}}\right\}^2.$$

Taking the logarithm of each member, we have by known formulæ,

$$\psi = \phi + 2\Sigma \text{ arc tan } \{\Delta . a . 2r\omega \tan\phi\} . r = 1, 2, \ldots \frac{n-1}{2} \; . . (E).$$

Let us now differentiate (E), remembering that

$$d\psi = dv\sqrt{(1 - h^2 \sin^2\psi)},\ d\phi = du\sqrt{(1 - k^2\sin^2\phi)};$$

and there results

$$dv\sqrt{(1 - h^2\sin^2\psi)} = du\sqrt{(1 - k^2\sin^2\phi)}$$
$$+ 2\,du\sqrt{(1 - k^2\sin^2\phi)} . \Sigma \frac{\Delta . a . 2r\omega}{1 - k^2 s^2 . a . 2r\omega \sin^2\phi}.$$

The last of (D) developed may be put under the following form:

$$\beta\sqrt{(1-h^2\sin^2\psi)}=\sqrt{(1-k^2\sin^2\phi)}$$
$$+2\sqrt{(1-k^2\sin^2\phi)}\,.\,\Sigma\,\frac{\Delta\,.\,a\,.\,2r\omega}{1-k^2s^2.a\,.\,2r\omega\sin^2\phi};$$

the last but one divided by the last, member by member, gives $\frac{dv}{\beta}=du$, or $dv=\beta du$; which proves the success of the transformation.

The transformation we have obtained diminishes the modulus. We may easily derive one from it which increases it; thus make $\sin\psi=\tan\tau\sqrt{(-1)}=i\tan\tau$, $\sin\phi=i\tan\phi$; and (B) becomes

$$\frac{d\tau}{\sqrt{(1-h'^2\sin^2\tau)}}=\frac{\beta d\phi}{\sqrt{(1-k'^2\sin^2\phi)}}\ldots\ldots(F).$$

Developing (C), and putting $s.a.v=\sin\psi=i\tan\tau$, $s.a.u=\sin\phi=i\tan\phi$; those formulæ will give $\sin\tau$, $\cos\tau$, &c. expressed in terms of ϕ.

Suppose now n infinite, $k=1$, $k'=0$; then $K=\infty$, $K'=\frac{\pi}{2}$.

Also (F) gives $H'=\beta K'=\frac{\beta\pi}{2}$, or $\beta=\frac{2H'}{\pi}$. Again, (B) gives $H=\beta\omega$, or $\omega=\frac{H}{\beta}=\frac{\pi H}{2H'}$. When $k=1$, $u=\int\frac{d\phi}{\cos\phi}=$ $\log\sqrt{\left(\frac{1+\sin\phi}{1-\sin\phi}\right)}$, which gives $\sin\phi=\frac{1-c^{-2u}}{1+c^{-2u}}$, c being the base of hyp. log. If for u in the last we put ω, 2ω, &c., or $\frac{\pi H}{2H'}$, $\frac{\pi H}{H'}$, &c. and for $\sin\phi$, $\sin am\,.\,\omega$, $\sin am\,.\,2\omega$, &c. we shall have these last quantities, and all other functions of $am.\omega$, expressed in terms of $c^{-\frac{\pi H}{H'}}$.

If we make $\tau=am\,.\,v'$, (F) becomes $v'=\beta\phi$; and $\sin\phi=\sin\frac{v'}{\beta}=\sin\frac{\pi v'}{2H'}$. Substituting these values in the formulæ obtained from (C) as above indicated, we shall have $\sin am\,.\,v'$, $\cos am\,.\,v'$ &c. expressed in terms of $\sin\frac{\pi v'}{2H'}$; or of the function itself. Thus the functions of the amplitude of v' may be expressed in terms of v' itself by infinite factorials or infinite series, as M. Jacobi has expressed them.

The transformation effected in this paper is that of M. Jacobi.

In page 47 of his work, changing his notation into mine where they differ, he has

$$s.a.v = \sqrt{\left(\frac{k^n}{h}\right)} s.a.u\ s.a\,(u + 4\omega) \ldots s.a\,\{u + 4\,(n-1)\,\omega\}.$$

The middle factor is $s\,.\,a\,.\,\{u + 2\,(n-1)\,\omega\}$; and the following factors easily reduce by (A) to

$$\pm\ s.a(u + 2\omega),\ \pm\ s.a(u + 6\omega), \text{ \&c.}$$

Also h' has the same value in this theory as λ' in Jacobi; see page 46. Therefore his transformation and mine are the same. I now proceed to point out an error into which he has fallen.

It has appeared that the factorial,

$$s.a.\omega\ s.a.3\omega\ s.a.5\omega \ldots s.a\,(2n-1)\,\omega$$
$$= \{s.\omega\ a.s.a.3\omega \ldots s.a\,(n-2)\,\omega\}^2.s.a.n\omega,$$

enters into the values of sin $am\,.\,v$, of h and h', and of β. If therefore $s.a.n\omega$ be nothing or infinite, it renders those values faulty. We have made $\omega = \frac{K}{n}$; but let $\omega = \frac{2K}{n}$; then sin $am\,.\,n\omega$ = sin $am\,.\,2K = 0$. In like manner, if m be any even integer, and $\omega = \frac{mK}{n}$, sin $am\,.\,n\omega$ = sin $am\,.\,mK = 0$.

We cannot therefore have $\omega = \frac{mK}{n}$, if m be an even integer.

Let $\sin\phi = \sqrt{(-1)}\,.\tan\psi = i\tan\psi$; see Jacobi, p. 34. Then

$$\frac{d\phi}{\sqrt{(1 - k^2 \sin^2\phi)}} = \frac{id\psi}{\sqrt{(1 - k'^2 \sin^2\psi)}}.$$

Whence the following,

$$\sin am\,(iu, k) = i \tan am\,(u, k');$$

$$\cos am(iu, k) = \frac{1}{\cos am(u, k')},\ \Delta am\,(iu, k) = \frac{\Delta am\,(u, k')}{\cos am(u, k')} \,.\,.(G).$$

Now suppose $\omega = \frac{iK'}{n}$. This is making m in the last case equal to nothing, which is an even number. But sin $am\,.\,n\omega$ = sin $am\,.\,iK'$, or = sin $am\,(iK', k) = i \tan am\,(K', k') = i\,\infty$. This must render all the quantities above mentioned faulty, especially as all the other factors which enter into their values, will be finite. This is the value of ω in one of M. Jacobi's transformations, which therefore must be faulty.

Suppose in general with M. Jacobi, that $\omega = \frac{mK + m'iK'}{n}$, where m and m' are any integers affirmative or negative, but having no common divisor which also measures n. If m be

even, $\sin am . mK = 0$, $\cos am . mK = \pm 1$; and by the first of (A), $\sin am . n\omega = \sin am (mK + im'K') = \pm \sin am (im'K') = \pm \sin am (im'K', k) = \pm i \tan am (m'K', k')$ by $(G) = 0$ or ∞, according as m' is even or odd. This must render faulty all the quantities into the values of which $\sin am . n\omega$ enters; for the other factors are neither nothing nor infinite. M. Jacobi therefore has erred in supposing that m may be even.

VI.—ON THE SOLUTION OF FUNCTIONAL DIFFERENTIAL EQUATIONS.

By R. L. Ellis, B.A. Fellow of Trinity College.

It is well known that the solution of a considerable class of differential equations may be effected by means of differentiation. Clairaut's equation is a particular case of this class. We will begin by considering it.

$$y = px + fp \dots (1) \qquad \left(p = \frac{dy}{dx}\right),$$

where f denotes any given function.

Differentiating (1), we get

$$(x + f'p)\, q = 0 \dots (2),$$

hence $\quad q = 0, \;$ or $\; x + f'p = 0 \dots (3).$

The first of these equations gives the complete integral. Being twice integrated it becomes $y = ax + b$; and on substitution in (1), we get $b = fa$, therefore $y = ax + fa \dots\dots (4)$ is the complete integral of (1).

It has always been supposed, in this and similar cases, that f must necessarily be a given function. But this condition is not essential: a differential equation, *e.g.* such as (1), will, when solved, give y as a function of x. Now the function f, which enters into (1) may, instead of being given, as is usually the case, be in some way dependent on the function which y is of x. Thus the form of f is unknown, until that of the latter function has been determined. It is evident that according to the classification proposed in the last number of the Journal, (1) is in all such cases a functional equation. For the unknown operation f is performed on p, which is itself the result of the unknown operation ψ' performed on x. (we suppose $y = \psi x$).

To differential functional equations, ordinary methods of solution do not, generally speaking, apply, because they require a knowledge of the forms of the functions on which they operate. But in the case before us, the differentiation and

subsequent substitution, by which (4) was derived from (1), are independent of any knowledge of the nature of f. Consequently (4) is always true.

Let us suppose, for instance, that $f = -m\psi$, m being a constant; then

$$\psi x - x\psi' x + m\psi\psi' x = 0 \ldots\ldots (5).$$

We are of course obliged to introduce a functional notation: (4) in this case becomes

$$\psi x = ax - m\psi a \ldots\ldots\ldots\ldots (6).$$

In order to determine ψa, put $x = a$;

$$\text{then} \quad \psi a = \frac{a^2}{1+m},$$

$$\text{and} \quad \psi x = ax - \frac{m}{1+m} a^2 \ldots . (7),$$

which is a solution of (5).

In the ordinary cases of Clairaut's equation, the factor $x + f'p = 0$ leads to the singular solution; and so it does when f is an unknown function.

Thus, in the example just considered, as $f' = -m\psi'$, we shall have

$$m\psi'\psi' x = x \ldots\ldots (8).$$

Of this a solution is

$$\psi' x = \frac{x}{\sqrt{(m)}}.$$

Hence we get, by integration,

$$\psi x = \frac{x^2}{2\sqrt{(m)}} + C.$$

On substitution it is found that $C = 0$, therefore

$$\psi x = \frac{x^2}{2\sqrt{(m)}} \ldots\ldots (9)$$

is a new solution of (5), and perfectly distinct from (7).

If $m = 1$, (5) and (7) become respectively

$$\psi x - x\psi' x + \psi\psi' x = 0 \ldots\ldots (5'),$$

$$\psi x = ax - \tfrac{1}{2} a^2 \ldots\ldots\ldots\ldots (7');$$

in this case, (8) admits of a variety of simple solutions. Thus we shall have

$$\psi x = \tfrac{1}{4} c^2 - \tfrac{1}{2}(c - x)^2,$$
$$\psi x = \tfrac{1}{2} + \log x,$$
$$\&c. = \&c.$$

as singular solutions of (5').

The preceding remarks are sufficient to indicate the existence of a class of functional equations, to which a considerable portion of the theory of singular solutions may be applied. They appear therefore to possess some interest with reference to this theory, independently of the method they suggest for the solution of such equations.

In fact the theory can hardly be considered complete, unless some notice is taken of the equations of which we have been speaking. They have been excluded from it, because the function f, which they involve, is not, as in the ordinary case, a known function. But this, it has been already remarked, is not an essential distinction.

On the other hand, the method by which the singular solution is in the common theory deduced from the complete integral, does not apply to the cases now considered. It appears unnecessary to point out the reason of this difference.

With regard to the class of differential equations, which, like Clairaut's, separate into factors on differentiation, we may refer to Lagrange's *Leçons sur le Calcul des Fonctions*, l. 16^{me}. He there shows that if a differential equation of the first order can be put into the form $M = fN$, where M and N are the values of a and b deduced from

$$F(xyab) = 0,$$

$$\frac{d}{dx} F(xyab) = 0;$$

then, when differentiated, it will resolve itself into two factors, one of which leads to the singular solution, and the other to the complete integral. (The latter is, as may readily be seen,

$$F(xyfb \,.\, b) = 0.)$$

The demonstration of this proposition is probably familiar to the majority of my readers, and I shall therefore not dwell upon it. Similar considerations apply to equations of higher orders.

Generalizing the remarks already made, we see that in the equation

$$M = fN,$$

the function f need not be a given one; it may be, in any way we please, dependent on the function which, in virtue of this equation, y is of x. In all such cases the equation in question is functional. Nevertheless, Lagrange's reasoning applies as much in these as in other cases. Let us take one or two examples of what has been said.

The following problem may be proposed.

Any point P of a certain curve is referred to the axis of x in M, and to that of y in N. MP is produced to Q; PQ is taken equal to a, and NQ touches the curve. Find its equation.

Let x, ψx be the co-ordinates of the point where NQ touches the curve.

$$ON = \psi x - x\psi' x. \quad NP = \frac{a}{\psi' x},$$

and as P is a point in the curve,

$$ON = \psi\,\{NP\}, \text{ or}$$

$$\psi x - x\psi' x = \psi\left(\frac{a}{\psi' x}\right) \ldots\ldots (10).$$

This is the equation of the problem. Differentiating it, we get

$$\psi'' x = 0,$$

$$x = \frac{a}{(\psi' x)^2}\,\psi'\left(\frac{a}{\psi' x}\right).$$

The former equation gives the complete integral, but, for a reason I shall hereafter notice, leads to no tangible solution of the problem; the latter corresponds to the singular solution.

In order to solve it, assume

$$\frac{a}{\psi' x} = \chi x;$$

$$\text{then} \quad \psi'\,\frac{a}{\psi' x} = \frac{a}{\chi^2 x} \text{ and } \frac{x}{\chi x} = \frac{\chi x}{\chi^2 x}.$$

Let $x = u_z$, $\chi x = u_{z+1}$, and therefore $\chi^2 x = u_{z+2}$.

Then $\dfrac{u_z}{u_{z+1}} = \dfrac{u_{z+1}}{u_{z+2}} = \dfrac{1}{C}$, where C is arbitrary;

$$\text{therefore } \chi x = Cx.$$

C is a function of z, which does not change when $z + 1$ is substituted for z.

We confine ourselves to the only simple case, that in which it is an absolute constant; then

$$\psi' x = \frac{a}{Cx} = \frac{b}{x} \ldots\ldots (bC = a)$$

and $\psi x = b \log \dfrac{x}{c}$ .... c being an arbitrary constant.

On substitution, we find $b = ae$; therefore

$$y = ae \log \frac{x}{c} \ldots\ldots (11)$$

is a solution of the problem.

This is the equation of a logarithmic curve, which has therefore the required property. The method employed to resolve the equation in χx, namely

$$x\chi^2 x = (\chi x)^2,$$

is applicable to every equation of the form

$$F(x \,.\, \chi x \,\ldots\, \chi^n x) = 0 \quad \ldots\ldots (12).$$

Every such equation may be at once reduced to the following equation in finite differences,

$$F(u_z\, u_{z+1} \,\ldots\, u_{z+n}) = 0 \quad \ldots\ldots (13).$$

This reduction is in reality a particular case of an important transformation due to Mr. Babbage, which often enables us to solve functional equations of the higher orders.

In (12) we may write for χx, $\phi f \phi^{-1} x$.

Hence $\chi^2 x = \phi f^2 \phi^{-1} x$ &c. = &c., and (12) becomes

$$F(\phi x \,.\, \phi f x \,\ldots\, \phi f^n x) = 0 \quad \ldots\ldots (14),$$

by putting ϕx for x; f being a known function, (14) is a functional equation of the first order.

Such is Mr. Babbage's method. Let $fx = 1 + x$; (14) becomes

$$F\{\phi x \,.\, \phi(1 + x) \,\ldots\, \phi(n + x)\} = 0,$$

and if we denote ϕx by u_x, and replace x by z, we shall obtain (13).

It must be admitted, that it is difficult to prove that the generality of 12) is not restricted by these transformations. They are however often useful, and serve to illustrate what was remarked in the last number, with respect to the affinity of functional equations, and equations in finite differences.

If, instead of (10), we had taken the more general equation

$$\psi x - x\psi' x = \psi\left(\frac{a}{\psi' x}\right) + A \quad \ldots\ldots (15),$$

where A is an arbitrary constant, precisely the same method would have applied. In this case the factor $\psi'' x = 0$ would have led to the result

$$\psi x = ax + \beta,$$

and by substitution $\quad \beta = a + \beta + A.$

$$\text{therefore} \quad a + A = 0, \quad \text{or } \beta = \infty.$$

Now in the case we have been considering, the former condition is not fulfilled; hence we must have $\beta = \infty$, and the geometrical interpretation of the complete integral is a right line at an infinite distance from the axis of abscissæ.

We not unfrequently meet with similar cases, in which the complete integral becomes nugatory or impossible in the process of introducing the necessary relation between its constants.

Under particular conditions, however, this difficulty does not occur, and then we obtain, what in the ordinary methods of discussing functional differential equations, appears to be a *conjugate* solution, unconnected with any other; (15) would be an instance of this, were $a + A = 0$.

I shall next consider a celebrated problem, first proposed by Euler, in the Petersburgh memoirs.

In a certain class of curves, the square of any normal exceeds the square of the ordinate drawn from its foot by a certain quantity a.

Let $y^2 = \psi x$ be the equation of the curve. The subnormal is therefore $\frac{1}{2}\psi' x$, and the equation of the problem consequently is

$$\psi(x + \tfrac{1}{2}\psi' x) = \psi x + \tfrac{1}{4}(\psi' x)^2 - a \; \ldots . \; (16).$$

Differentiating this, we get

$$\psi'(x + \tfrac{1}{2}\psi' x) = \psi' x;$$

$$\text{or} \quad 1 + \tfrac{1}{2}\psi'' x = 0.$$

The first of these two equations leads to the singular solutions. In order to solve it, let

$$x + \tfrac{1}{2}\psi' x = \chi x,$$

$$\text{then} \quad \chi^2 x - 2\chi x + x = 0.$$

Hence by the transformation already noticed,

$$u_{z+2} - 2u_{z+1} + u_z = 0,$$

$$\text{whence} \quad u_z = Pz + zP_1 z,$$

where Pz and $P_1 z$ are functions of z, which remain unchanged when z increases by unity;

$$\text{therefore} \quad u_{z+1} = Pz + (z+1)P_1 z.$$

Hence we have

$$\left.\begin{aligned} \tfrac{1}{2}\psi' x &= P_1 z \\ x &= Pz + zP_1 z \end{aligned}\right\} \text{for the required solution.}$$

$$dx = (P'z + P_1 z + zP_1' z)\,dz;$$

$$\text{therefore} \quad y\,dy = P_1 z\,(P'z + P_1 z + zP_1' z)\,dz;$$

and integrating by parts, we get

$$\left.\begin{aligned} y^2 &= P_1 z\,(2Pz + zP_1 z) + \int (P_1 z)^2\,dz \\ x &= Pz + zP_1 z \end{aligned}\right\} \ldots . \; (17),$$

for a general solution of the proposed problem. (The parameter a is involved in $P_1 z$).

Let us suppose Pz and $P_1 z$ constant;

$$y^2 = a(2b + az) + a^2 z + C,$$

$$x = b + az;$$

$$\text{therefore} \quad y^2 = 2ax + C.$$

On substitution we find

$$a^2 = -a.$$

Thus, in order to a real result, we must suppose a negative, *e.g.* let $a = -k^2$; then

$$y^2 = 2kx + C \ldots\ldots (18),$$

the equation to a parabola, which accordingly is a solution of the problem, and the only simple one it admits of.

When $a = 0$, it becomes two straight lines parallel to the axis.

The other factor $1 + \frac{1}{2}\psi', x = 0$ gives, on integration,

$$\psi x + x^2 = \alpha x + \beta \ldots\ldots (19),$$

the equation to a circle; but on substitution we find

$$\tfrac{1}{4}a^2 = \tfrac{1}{4}a^2 - a,$$

which leads to no result, unless $a = 0$.

A solution of this problem, by Poisson, is given at p. 591 of the last volume of Lacroix's great work. It is apparently equivalent in point of generality to (17); and the author points out its incompleteness in the case of $a = 0$. The preceding views show distinctly the nature and origin of the new solution which then presents itself. Mr. Babbage also has considered this problem at the end of his second essay on the Calculus of Functions (vide *Phil. Trans.* 1816, p. 253). But I believe it will be found that his solution is erroneous.

Notwithstanding the length this paper has already reached, I must endeavour to point out, as briefly as possible, my reasons for thinking so.

Mr. Babbage, confines himself to the case of $a = 0$. He begins by demonstrating the existence of a relation, equivalent, excepting a difference of notation, to $\psi'\{x + \frac{1}{2}\psi'x\} = \psi'x$, but in doing this, loses sight of the other factor $1 + \frac{1}{2}\psi''x = 0$.

This relation shows that $\psi'x$ is constant, for a series of points in the curve, and therefore, Mr. Babbage reasons, we may consider it as a constant in (16), which thus becomes an equation in finite differences. He integrates it on this supposition, and adds an arbitrary function of $\psi'x$, which has been treated as an absolute constant. The result is therefore

$$\psi x = \tfrac{1}{2}x\psi'x + f.\tfrac{1}{2}\psi'x,$$

$$\text{or} \quad y^2 = xy\frac{dy}{dx} + f\left(y\frac{dy}{dx}\right) \ldots\ldots (20),$$

which is an ordinary differential equation.

This process appears to have been suggested by an incorrect analogy with the way in which arbitrary functions are introduced into partial differential equations.

A little consideration would have convinced Mr. Babbage, that by integrating (16) as an equation in finite differences, he only passed discontinuously from one ordinate of the curve to another, and therefore could not obtain a continuous relation between x and y. The legitimate result of his process is merely,

$$\psi\{x + \tfrac{1}{2}x\psi' x\} - \psi x = \frac{n}{4}(\psi' x)^2,$$

where n is any positive or negative integer. This is quite different from

$$\psi x = \tfrac{1}{2}x\psi' x + f\{\tfrac{1}{2}\psi' x\}.$$

In exemplifying equation (20), Mr. Babbage first supposes

$$f\left(y\frac{dy}{dx}\right) = \infty \times y\frac{dy}{dx},$$

and thus obtains the equation of a straight line parallel to Ox, as a solution of the problem, which undoubtedly it is.

In his next example $f\left(y\dfrac{dy}{dx}\right) = a^2$. By making the constant of integration imaginary, he gets $y^2 = a^2 - x^2$, the equation to a circle. But although this is also a real solution, it has no connection with the relation $\psi'\{x + \frac{1}{2}\psi' x\} = \psi' x$, from which it appears to be derived. It is, as we have seen, a particular case of the complete integral. Consequently if the method pursued had been correct, it could not have given this solution.

The preceding pages appear to contain the germ of a general theory of differential functional equations; a subject of great extent, and ultimately, perhaps, of considerable importance. But it cannot be denied, that hitherto the Calculus of Functions has not led to many results of much interest. Its value arises chiefly from the wide views it gives of the science of the combination of symbols.

VII.—ON CERTAIN DEFINITE INTEGRALS.

By Arthur Cayley, B.A. Trinity College.

In the first place, we shall consider the integral

$$V = \iint \ldots (n \text{ times}) \frac{dx\,dy\ldots}{\{(a-x)^2 + (b-y)^2\ldots\}^{\frac{n}{2}-1}},$$

the integration extending to all real values of the variables, subject to the condition

$$\frac{x^2}{h^2}+\frac{y^2}{h_1^2}+\ldots < \text{ or } = 1,$$

and the constants a, b, &c. satisfying the condition

$$\frac{a^2}{h^2}+\frac{b^2}{h_1^2}\ldots > 1.$$

We have

$$\frac{dV}{da} = -(n-2)\iint\ldots(n \text{ times}).\frac{(a-x)\,dx\,dy\ldots}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}n}},$$

$$= -(n-2)\,\frac{2hh_{,}\ldots\pi^{\frac{1}{2}n}a}{\sqrt{(\xi+h^2)}\Gamma(\frac{1}{2}n)}\int_0^1\frac{x^{n-1}.\,dx}{[\{\xi+h^2+(h_1^2-h^2)x^2\}\{\xi+h^2+(h_{,,}^2-h^2)x^2\}\ldots]^{\frac{1}{2}}},$$

ξ being determined by the equation

$$\frac{a^2}{\xi+h^2}+\frac{b^2}{\xi+h_{,}^2}\ldots = 1,$$

by a formula in a paper, "On the Properties of a Certain Symbolical Expression," in the preceding number of this Journal. (ξ having been substituted for η^2.)

Let the variable x, on the second side of the equation, be replaced by ϕ, where

$$x^2 = \frac{\xi+h^2}{\xi+h^2+\phi};$$

we have, without difficulty,

$$\frac{dV}{da} = -(n-2).\frac{hh_{,}\ldots\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}.a\int_0^\infty\frac{d\phi}{(\xi+h^2+\phi)\sqrt{\Phi}},$$

where $\Phi = (\xi+h^2+\phi)(\xi+h_{,}^2+\phi)\ldots.$

and similarly,

$$\frac{dv}{db} = -(n-2).\frac{hh_1\ldots\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}b.\int_0^\infty\frac{d\phi}{(\xi+h_{,}^2+\phi)\sqrt{\Phi}},$$

&c. . . . . .

From these values it is easy to verify the equation

$$V = \frac{(n-2)\,hh_{,}\ldots\pi^{\frac{1}{2}n}}{2\,\Gamma(\frac{1}{2}n)}\int_0^\infty\left(1-\frac{a^2}{\xi+h^2+\phi}-\frac{b^2}{\xi+h_{,}^2+\phi}\ldots\right).\frac{d\phi}{\sqrt{\Phi}}.$$

For this evidently verifies the above values of $\frac{dV}{da}$, $\frac{dV}{db}$, &c. if only the term $\frac{dV}{d\xi}d\xi$ vanishes.

$$\text{Now } \frac{dV}{d\xi} = \frac{(n-2)\,hh_{,}\ldots\pi^{\frac{1}{2}n}}{2\Gamma(\frac{1}{2}n)}\int_0^\infty d\phi.\frac{d}{d\xi}.\left(1-\frac{a^2}{\xi+h^2+\phi}\ldots\right).\frac{1}{\sqrt{\Phi}}.$$

Or, observing that

$$\frac{d}{d}\left(1-\frac{a^2}{\xi+h^2+\phi}-\dots\right)\frac{1}{\sqrt{(\Phi)}}=\frac{d}{d\phi}\left(1\ \frac{a^2}{\xi+h^2+\phi}\ \dots\right)\frac{1}{\sqrt{(\Phi)}},$$

and taking the integral from 0 to ∞,

$$\frac{dV}{d\xi}=-\frac{(n-2)hh_{,}\dots\pi^{\frac{1}{2}n}}{2\Gamma(\frac{1}{2}n)}\cdot\left(1-\frac{a^2}{\xi+h^2}-\frac{b^2}{\xi+h_{,}^2}\dots\right)\frac{1}{\sqrt{\{(\xi+h^2)(\xi+h_{,}^2)\}}}=0,$$

in virtue of the equation which determines ξ.

No constant has been added to the value of V, since the two sides of the equation vanish as they should do, for a, b... infinite, for which values ξ is also infinite and the quantity

$$\left(1-\frac{a^2}{\xi+h^2+\phi}\dots\right)\cdot\frac{1}{\sqrt{(\Phi)}},$$

which is always less than $\frac{1}{\sqrt{(\Phi)}}$, vanishes. Hence, restoring the values of V and Φ,

$$\iint\dots(n\text{ times})\ \frac{dx\,dy\dots}{\{(a-x)^2+(b-y)^2\dots\}^{\frac{1}{2}n-1}}$$

$$=\frac{(n-2)hh_{,}\dots\pi^{\frac{1}{2}n}}{2\Gamma(\frac{1}{2}n)}\cdot\int_0^\infty\left(1-\frac{a^2}{\xi+h^2+\phi}-\frac{b^2}{\xi+h_{,}^2+\phi}\dots\right)\cdot\frac{d\phi}{\sqrt{\{(\xi+h^2+\phi)(\xi+h_{,}^2+\phi)\}}}$$

.... the limits of the first side of the equation, and the condition to be satisfied by a, b, &c., also the equation for the determination of ξ, as above.

The integral

$$V'=\iint\dots(n\text{ times})\cdot\frac{dx\,dy\dots}{\{(a-x)^2\dots\}^{\frac{1}{2}n}},$$

between the same limits, and with the same condition to be satisfied by the constants, has been obtained in the paper already quoted. Writing ξ instead of η^2, and $x^2=\frac{\xi}{\xi+\phi}$, we have

$$V'=\frac{hh_{,}\dots\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}\int_0^\infty\frac{d\phi}{\sqrt{\{(\xi+\phi)(\xi+h^2+\phi)(\xi+h_{,}^2+\phi)\}\dots}},$$

$$\text{where}\qquad \frac{a^2}{\xi+h^2}+\frac{b^2}{\xi+h_{,}^2}\dots=1.$$

Let $\nabla=\frac{d^2}{da^2}+\frac{d^2}{db^2}+\dots$ Then by the assistance of a formula,

$$\nabla^q \cdot \frac{1}{(a^2+b^2\ldots)^i}$$
$$= 2i.(2i+2)\ldots(2i+2q-2)(2i+2-n)\ldots(2i+2q-n).\frac{1}{(a^2+b^2\ldots)^{i+q}}$$

given in the same paper, in which it is obvious that $a, b\ldots$ may be changed into $a-x, b-y$, &c. . . . : also putting $i=\frac{1}{2}n$, we have

$$\iint\ldots(n \text{ times}) \cdot \frac{dx\,dy\ldots}{\{(a-x)^2+.\ \ldots\}^{\frac{1}{2}n+q}}$$
$$= \frac{hh_{,}\ \ldots\ldots\ \pi^{\frac{1}{2}n}}{2^{2q}\,.\,1\,.\,2\ldots q\,.\,\Gamma(\frac{1}{2}n+q)}\int_0^\infty d\phi \,.\, \nabla^q \,.\, \frac{1}{\sqrt{\{(\xi+\phi)(\xi+h^2+\phi)\ldots}}$$

Now in general

$$\nabla\chi\xi = \chi'\xi \,.\left(\frac{d^2\xi}{da^2}+\frac{d^2\xi}{db^2}\ \ldots\ldots\right)+\chi''\xi \,.\left\{\left(\frac{d\xi}{da}\right)^2+\left(\frac{d\xi}{db}\right)^2\ldots\ldots\right\}$$
$$= \chi'\xi \,.\, \Sigma\left(\frac{d^2\xi}{da^2}\right)+\chi''\xi \,.\, \Sigma\left(\frac{d\xi}{da}\right)^2, \text{ suppose.}$$

$$\text{But} \qquad \Sigma\frac{a^2}{(\xi+h^2)} = 1.$$

$$\text{Hence} \qquad \frac{2a}{\xi+h^2}-\left\{\Sigma\frac{a^2}{(\xi+h^2)^2}\right\} \cdot \frac{d\xi}{da} = 0.$$

$$\text{Whence} \qquad \Sigma\left(\frac{d\xi}{da}\right)^2 = \frac{4}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}}.$$

Also

$$\frac{2}{\xi+h^2}-4\frac{a}{(\xi+h^2)^2}\cdot\frac{d\xi}{da}+2\left\{\Sigma\frac{a^2}{(\xi+h^2)^3}\right\}\left(\frac{d\xi}{da}\right)^2-\left\{\Sigma\frac{a^2}{(\xi+h^2)^2}\right\}\frac{d^2\xi}{da^2}=0.$$

Whence taking the sum Σ, and observing that

$$-4\,\Sigma\frac{a}{(\xi+h^2)^2}\cdot\frac{d\xi}{da} = -8\,.\,\frac{\Sigma\dfrac{a^2}{(\xi+h^2)^3}}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}} = -2\,\Sigma\frac{a^2}{(\xi+h^2)^3}\cdot\Sigma\left(\frac{d\xi}{da}\right)^2,$$

$$2\Sigma\frac{1}{\xi+h^2}-\left\{\Sigma\frac{a^2}{(\xi+h^2)^2}\right\}.\,\Sigma\left(\frac{d^2\xi}{da^2}\right)=0\,;$$

$$\text{or} \qquad \Sigma\left(\frac{d^2\xi}{da^2}\right) = \frac{2\Sigma\dfrac{1}{\xi+h^2}}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}}.$$

$$\text{Or}\quad \nabla\chi\xi = \frac{2\chi'\xi\,.\,\Sigma\left(\dfrac{1}{\xi+h^2}\right)+4\chi''\,.\,\xi}{\Sigma\,\dfrac{a^2}{(\xi+h^2)^2}}.$$

Hence the function

$$\int_0^\infty d\phi\,.\,\nabla\,.\,\frac{1}{\sqrt{\{(\xi+\phi)(\xi+h^2+\phi)\}\ldots}}$$

(observing that differentiation with respect to ξ is the same as differentiation with respect to ϕ) becomes integrable, and taking the integral between the proper limits, its value is

$$-\frac{2\chi_0\xi\,.\,\Sigma\,\dfrac{1}{\xi+h^2}+4\chi_0'\,.\,\xi}{\Sigma\,\dfrac{a^2}{(\xi+h^2)^2}}.$$

$$\text{Where}\quad \chi_0\xi = \frac{1}{\sqrt{\{\xi\,.\,(\xi+h^2)(\xi+h_1^2)\}\ldots}}.$$

We have immediately

$$\frac{\chi_0'\,.\,\xi}{\chi_0\xi} = -\tfrac{1}{2}\,.\left(\frac{1}{\xi}+\Sigma\,\frac{1}{\xi+h^2}\right);$$

$$\text{or}\quad 2\chi_0\xi\,.\,\Sigma\left(\frac{1}{\xi+h^2}\right)+4\chi_0'.\xi = -2\,\frac{\chi_0\xi}{\xi}.$$

Whence

$$\int_0^\infty d\phi.\nabla.\frac{1}{\sqrt{\{(\xi+\phi)(\xi+h^2+\phi)\}}} = \frac{2}{\xi\sqrt{\xi\,(\xi+h^2)}\left\{\dfrac{a^2}{(\xi+h^2)^2}+\dfrac{b^2}{(\xi+h_1^2)^2}\ldots\right\}}.$$

And hence restoring the value of ∇, and of the first side of the equation,

$$\iint\ldots\ldots(n\text{ times})\,.\,\frac{dxdy\ldots\ldots}{\{(a-x)^2+\ldots\ldots\}^{\frac{1}{2}n+q}}$$

$$= \frac{hh_1\ldots\ldots\pi^{\frac{1}{2}n}}{2^{2q-1}\,1\,.\,2\ldots q\Gamma(\frac{1}{2}n+q)}\left(\frac{d^2}{da^2}+\frac{d^2}{db^2}\ldots\ldots\right)^{q-1}$$

$$\frac{1}{\xi\sqrt{\{\xi(\xi+h^2)(\xi+h_1^2)\ldots\}}\left\{\dfrac{a^2}{(\xi+h^2)^2}+\dfrac{b^2}{(\xi+h_1^2)^2}+\ldots\ldots\right\}},$$

with the condition $\dfrac{a^2}{\xi+h^2}+\dfrac{b^2}{\xi+h_1^2}\ldots\ldots=1$;

from which equation the differential coefficients of ξ, which enter into the preceding result, are to be determined.

In general if u be any function of ξ, a, b ....

$$\left(\frac{d^2}{da^2}+\frac{d^2}{db^2}\cdots\right)u=\frac{4\,\dfrac{d^2u}{d\xi^2}+2\,\dfrac{du}{d\xi}\,.\,\Sigma\dfrac{1}{\xi+h^2}+4\Sigma\,\dfrac{a}{\xi+h^2}\,.\,\dfrac{d^2u}{d\xi da}}{\Sigma\,\dfrac{a^2}{(\xi+h^2)^2}}+\Sigma\frac{d^2u}{da^2},$$

from which the values of the second side for $q=1$, $q=2$, &c. may be successively calculated.

The performance of the operation $\left(\frac{d}{da}\right)^p\left(\frac{d}{db}\right)^q\left(\frac{d}{dc}\right)^r$, upon the integral V', leads in like manner to a very great number of integrals, all of them expressible algebraically, for a single differentiation, renders the integration with respect to ϕ possible. But this is a subject which need not be further considered at present.

We shall consider, lastly, the definite integral

$$U=\iint\ldots(n\text{ times})\,.\,\frac{(a-x)f\left(\dfrac{x^2}{h^2}+\dfrac{y^2}{h_1^2}+\right)dxdy\ldots}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}n}},$$

limits, &c. as before. This is readily deduced from the less general one

$$\iint\ldots(n\text{ times})\,.\,\frac{(a-x)\,dxdy\,\ldots}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}n}}.$$

For representing this quantity by $F.(h, h_1\ldots)$, it may be seen that

$$U=\int_0^1 f(m^2)\,.\,\frac{d}{dm}\,F.(mh, mh_1\ldots)\,dm.$$

Now in the value of $F.(h, h_1\ldots)$, changing $h, h_1\ldots$ into $mh, mh_1\ldots$ also writing $m^2\phi$ instead of ϕ, and $m^2\xi'$ for ξ, we have

$$F.(mh, mh_1\ldots)=\frac{hh_1\ldots\pi^{\frac{1}{2}n}}{\Gamma\left(\frac{1}{2}n\right)}\,a\,.\int_0^\infty\frac{d\phi}{(\xi'+h^2+\phi)\sqrt{(\Phi')}}.$$

Where $\qquad \Phi'=(\xi'+h^2+\phi)(\xi'+h_1^2+\phi)\ldots$

And $\qquad \dfrac{a^2}{\xi'+h^2}+\dfrac{b^2}{\xi'+h_1^2}+\ldots.=m^2.$

Hence $\qquad \dfrac{d}{dm}\,.\,F.(mh, mh_1\ldots)=\dfrac{d\xi'}{dm}\,\dfrac{d}{d\xi'},\,F.(mh, mh_1\ldots),$

$$=\frac{hh_1\ldots\pi^{\frac{1}{2}n}}{\Gamma\left(\frac{1}{2}n\right)}\,a\,.\,\frac{d\xi'}{dm}\,.\int_0^\infty d\phi\,.\,\frac{d}{d\xi'}\,\frac{1}{(\xi'+h^2+\phi)\sqrt{(\Phi')}},$$

or, observing that $\frac{d}{d\xi'}$ is equivalent to $\frac{d}{d\phi}$, and effecting the integrations between the proper limits,

$$\frac{d}{dm}\Gamma.(mh, mh_{,}...) = -\frac{hh_{,}\ ...\ \pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}a.\frac{1}{(\xi'+h^2)\sqrt{\{(\xi'+h^2)(\xi'+h_{,}^2)\}}}\cdots$$

Substituting this value, also $f\left\{\frac{a^2}{\xi'+h^2}+\frac{b^2}{\xi'+h_1^2}+..\right\}$ for $f(m^2)$, in the value of U, and observing that $m=0$ gives $\xi'=\infty$, $m=1$, $\xi'=\xi$, where ξ is a quantity determined as before by the equation

$$\frac{a^2}{\xi+h^2}+\frac{b^2}{\xi+h_{,}^2}+\ldots.=1,$$

we have

$$U=-\frac{hh_{,}...\pi^{\frac{1}{2}n}a}{\Gamma(\frac{1}{2}n)}.\int_{\infty}^{\xi}\frac{f\left\{\frac{a^2}{\xi'+h^2}+\ldots.\right\}d\xi'}{(\xi'+h^2)\sqrt{\{(\xi'+h^2)(\xi'+h_{,}^2)\ldots\}}},$$

or writing $\phi+\xi$ for ξ', $d\xi'=d\phi$, the limits of ϕ are $0, \infty$; or, inverting the limits and omitting the negative sign,

$$U=\frac{hh_{,}...\pi^{\frac{1}{2}n}.a}{\Gamma(\frac{1}{2}n)}.\int_0^{\infty}\frac{f\left\{\frac{a^2}{\xi+h^2+\phi}+\frac{b^2}{\xi+h_{,}^2+\phi}+\ldots.\right\}d\phi}{(\xi+h^2+\phi)\sqrt{\{(\xi+h^2+\phi)(\xi+h_{,}^2+\phi)\ldots\}}};$$

which, in the particular case of $n=3$, may easily be made to coincide with known results. The analogous integral

$$\iint\ldots(n\text{ times}).\frac{f\left\{\frac{x^2}{h^2}+\frac{y^2}{h_{,}^2}+\ldots.\right\}dx\,dy\ldots.}{\{(a-x)^2+(b-y)^2\ldots.\}^{\frac{1}{2}n}}$$

is apparently not reducible to a single integral.

VIII.—DEMONSTRATIONS OF SOME GEOMETRICAL THEOREMS.

The following geometrical theorems may perhaps be interesting to some of the readers of the *Mathematical Journal.* They are founded on the decomposition into quadratic factors of the trinomial $x^{2n}-2a^nx^n\cos\theta+a^{2n}$, and are therefore intimately related to the well-known theorem of Cotes. We assume then the theorem

$$x^{2n}-2a^nx^n\cos\theta+a^{2n}=(x^2-2ax\cos\frac{\theta}{n}+a^2)(x^2-2ax\cos\frac{\theta+2\pi}{n}+a^2)\ldots$$

$$\{x^2-2ax\cos\frac{\theta+(n-1)\pi}{n}+a^2\}\ldots.\ (A).$$

whence also, by making $x = a$, and writing $n\phi$ for $\frac{\theta}{2}$, we have the known theorem

$$\sin n\phi = 2^{n-1} \sin \phi \sin\left(\phi + \frac{\pi}{n}\right) \sin\left(\phi + \frac{2\pi}{n}\right) \ldots \sin\left(\phi + \frac{n-1}{n}\pi\right) \ldots \text{(B)}.$$

If a regular polygon of $2n$ sides circumscribe a circle, and if $p_1\, p_2 \ldots p_{2n-1}\, p_{2n}$ be the perpendiculars drawn on its sides from any point in the circumference of the circle, then

$$p_1 p_3 \cdots p_{2n-1} + p_2 p_4 \cdots p_{2n} = \frac{r^n}{2^{n-2}},$$

where r is the radius of the circle.

Let the arc between the assumed point and the adjacent point of contact subtend an angle ϕ at the centre; then

$$p_1 = r(1 - \cos\phi) = 2r \sin^2 \frac{\phi}{2}$$

$$p_2 = r\left\{1 - \cos\left(\phi + \frac{2\pi}{2n}\right)\right\} = 2r \sin^2\left(\frac{\phi}{2} + \frac{\pi}{2n}\right),$$

$$p_3 = 2r \sin^2\left(\frac{\phi}{2} + \frac{\pi}{n}\right),$$

$$p_4 = 2r \sin^2\left(\frac{\phi}{2} + \frac{3\pi}{2n}\right) \text{ \&c. \&c.}$$

Hence $p_1 p_3 \ldots p_{2n-1} = 2^n r^n \sin^2 \frac{\phi}{2} \sin^2\left(\frac{\phi}{2} + \frac{\pi}{n}\right) \ldots \sin^2\left(\frac{\phi}{2} + \frac{n-1}{n}\pi\right)$,

and by the theorem (*B*), writing $\frac{1}{2}\phi$ for ϕ we have

$$p_1 p_3 \ldots p_{2n-1} = \frac{r^n}{2^{n-2}} \sin^2 \frac{n\phi}{2}.$$

In the same way we find

$$p_2 p_4 \cdots p_{2n} = \frac{r^n}{2^{n-2}} \sin^2 n\left(\frac{\phi}{2} + \frac{\pi}{2n}\right) = \frac{r^n}{2^{n-2}} \cos^2 n\frac{\phi}{2},$$

and therefore

$$p_1 p_3 \cdots p_{2n-1} + p_2 p_4 \cdots p_{2n} = \frac{r^n}{2^{n-2}} \ldots\ldots\ldots\ldots (1).$$

Also, by multiplying the preceding expressions together,

$$p_1 p_2 \cdots p_{2n-1} p_{2n} = \frac{r^{2n}}{2^{2n-2}} \sin^2 n\phi \ldots\ldots\ldots\ldots (2).$$

If the given point be within the circumference of the circle, and if c be its distance from the centre, and ϕ the angle which c makes with the radius drawn to the adjacent point of contact, we have

$$p_1 = r - c\cos\phi = x^2 - 2xy\cos\phi + y^2,$$

if $\quad x^2 + y^2 = r$, and $2xy = c$.

R

There are corresponding expressions for the other perpendiculars, and therefore

$$p_1 p_3 \dots p_{2n-1} = (x^2 - 2xy \cos \phi + y^2) \dots \left\{x^2 - 2xy \cos\left(\phi + \frac{n-1}{n}\pi\right) + y^2\right\}$$

$$= x^{2n} - 2x^n y^n \cos n\phi + y^{2n} \text{ by } (A).$$

Similarly

$$p_2 p_4 \dots p_{2n} = x^{2n} - 2x^n y^n \cos(n\phi + \pi) + y^{2n} = x^{2n} + 2x^n y^n \cos n\phi + y^{2n};$$

hence by subtraction,

$$p_2 p_4 \dots p_{2n} - p_1 p_3 \dots p_{2n-1} = 4x^n y^n \cos n\phi = \frac{c^n}{2^{n-2}} \cos n\phi \dots \dots (3).$$

If from the given point within the circumference, we draw perpendiculars $q_1\ q_2 \dots q_{2n}$ on the radii drawn to the points of contact, we have

$$q_1 = c \sin \phi,\ q_2 = c \sin\left(\phi + \frac{\pi}{n}\right) \dots q_{2n} = c \sin\left(\phi + \frac{2n-1}{n}\pi\right),$$

and therefore, considering magnitude only,

$$q_1 q_2 \dots q_{2n} = \frac{c^{2n}}{2^{2n-2}} \sin^2 n\phi \dots \dots \dots \dots (4).$$

Mr. Leslie Ellis has shown (*Journal*, Vol. II. p. 272,) that, if $f(\phi)$ be a rational and integral function of $\sin\phi$ and $\cos\phi$, in which the highest power of these quantities is r,

$$f(\phi) + f\left\{\phi + \frac{2\pi}{n}\right\} \dots + f\left\{\phi + \frac{n-1}{n}2\pi\right\} = \frac{n}{2\pi}\int_0^{2\pi} fx\, dx,$$

when $n > r$. By similar reasoning we may show that, when $n = r$,

$$f(\phi) + \dots + f\left\{\phi + \frac{n-1}{n}2\pi\right\}$$

$$= \frac{n}{2\pi}\int_0^{2\pi} f(x)\, dx + \frac{n}{\pi}\int_0^{2\pi} dx\, f(x) \cos nx \,.\, \cos n\phi,$$

$$+ \frac{n}{\pi}\int_0^{2\pi} dx\, f(x) \sin nx \,.\, \sin n\phi.$$

Now if $p_1 p_2 \dots p_n$ be the perpendiculars drawn from any point in the circumference of a circle, on a regular circumscribing polygon of n sides, we have

$$p_1^n = r^n (1 - \cos\phi)^n,\ p_2^n = r^n\left\{1 - \cos\left(\phi + \frac{2\pi}{n}\right)\right\}^n \text{ \&c.}$$

Therefore

$$\frac{n}{2\pi}\int_0^{2\pi} dx\, f(x) = r^n \int_0^{2\pi} dx\,(1 - \cos x)^n = \frac{(2n-1)(2n-3)\dots 3\,.\,1}{(n-1)(n-2)\dots 2\,.\,1} r^n,$$

$$\frac{n}{\pi}\int_0^{2\pi} dx\, f(x) \cos nx = \frac{n}{\pi} r^n \int_0^{2\pi} dx\,(1 - \cos x)^n \cos nx = (-)^n \frac{n}{2^{n-1}}$$

$$\text{and} \quad \frac{n}{\pi}\int_0^{2\pi} dx\,(1-\cos nx)^n \sin nx = 0.$$

These results are easily arrived at, by observing that

$$(1-\cos x)^n = (-)^n \{(\cos x)^n - n(\cos x)^{n-1} + \&c.\},$$

$$= (-)^n \left\{\frac{\cos nx}{2^{n-2}} - \&c.\right\}$$

Hence,

$$\Sigma(p^n) = r^n \left\{\frac{(2n-1)(2n-3)\ldots\ldots 3\,.\,1}{(n-1)(n-2)\ldots\ldots 2\,.\,1} + (-)^n \frac{n}{2^{n-1}} \cos n\phi\right\};$$

but in the same manner as in (1), we find

$$p_1\, p_2 \ldots p_n = \frac{r^n}{2^{n-2}} \sin^2 n\,\frac{\phi}{2} = \frac{r^n}{2^{n-1}}(1-\cos n\phi).$$

Therefore

$$\Sigma(p^n) + (-)^n p_1 p_2 \ldots p_n = r^n \left\{\frac{(2n-1)(2n-3)\ldots 3.1}{(n-1)(n-2)\ldots 2.1} + (-)^n \frac{n}{2^{n-1}}\right\} \ldots (5).$$

In the same way, if c_1, c_2, &c. be the chords drawn from the given point to the angle of a regular inscribed polygon, we have

$$c_1^2 = 2r^2(1-\cos\phi), \ \&c.$$

and therefore

$$\Sigma(c^{2n}) + (-)^n (c_1, c_2 \ldots\ldots c_n)^2$$

$$= r^n \left\{\frac{2^n(2n-1)(2n-3)\ldots\ldots 3.1}{(n-1)(n-2)\ldots\ldots 2.1} + (-)^n 2n\right\} \ldots\ldots (6).$$

Theorems of this kind are not confined to the circle: thus, in the parabola, if n tangents be drawn such that the arcs between the points of contact subtend equal angles at the focus, and if $p_1, p_2 \ldots\ldots p_n$, be the perpendiculars on them from the focus, we shall have

$$p_1 p_2 \ldots\ldots p_n = 2^{n-1} a^n \operatorname{cosec} \frac{n\theta}{2},$$

a being one-fourth of the parameter, and θ being the angle which the axis of the parabola makes with the radius vector drawn to the adjacent point of contact. For the equation to the parabola being

$$\frac{1}{r} = \frac{1-\cos\theta}{2a} = \frac{1}{a}\sin^2\frac{\theta}{2},$$

$$\text{we have} \quad \frac{1}{p_1^2} = \frac{1}{ar} = \frac{1}{a^2}\sin^2\frac{\theta}{2},$$

$$\text{or} \quad \frac{1}{p_1} = \frac{1}{a}\sin\frac{\theta}{2},$$

and similarly for the others. Hence

$$\frac{1}{p_1 . p_2 \ldots . p_n} = \frac{1}{a^n} \sin\frac{\theta}{2} \sin\left\{\frac{\theta}{2} + \frac{\pi}{n}\right\} \ldots . \sin\left\{\frac{\theta}{2} + \frac{n-1}{n}\pi\right\}$$

$$= \frac{1}{2^{n-1}a^n} \sin\frac{n\theta}{2};$$

and therefore $p_1 . p_2 \ldots . p_n = 2^{n-1}a^n \operatorname{cosec}\frac{n\theta}{2} \ldots . (7).$

The theorems (1) and (5) were discovered by Professor Wallace, of Edinburgh, about the year 1791, but they have not, we believe, been as yet published.

D.

IX.—ON THE MOTION OF A SOLID BODY ABOUT ITS CENTRE OF GRAVITY.

I HAVE attempted in this paper the solution of the problem of the motion of a rigid body about its centre of gravity, fixed and acted on by no forces, by at once making use of the general dynamical principles, viz. those of the Conservation of *Vis Viva*, and of the Conservation of Areas, which furnish two of the integrals of the equation of motion. And though the final result thus obtained is not new, yet it is interesting to see the previous solution thus verified. I shall adopt the usual notation. The origin of the co-ordinates is the centre of gravity $\left.\begin{matrix} xyz \\ x_1\, y_1\, z_1 \end{matrix}\right\}$ the co-ordinates of an element δm of the body referred to fixed axes in space, and to the principal axes in the body respectively; ABC the moments of inertia about the principal axes $\omega_1\ \omega_2\ \omega_3$ the angular velocities $\left.\begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{matrix}\right\}$ the cosines of the inclinations of $x_1\ y_1\ z_1$ to $\begin{cases} x \\ y \\ z \end{cases}$

The *vis viva* of the body (since the motion is wholly of rotation about the centre of gravity) is

$$\Sigma\delta m \left\{\omega'^2 (y^2 + z^2) + \omega''^2 (x^2 + z^2) + \omega'''^2 (x^2 + y^2)\right\},$$

$\omega'\ \omega''\ \omega'''$ having references to the fixed axes; and

$\left.\begin{matrix} y^2 + z^2 \\ z^2 + x^2 \\ x^2 + y^2 \end{matrix}\right\}$ being the squares of the distances of the elements δm from the axes of $\begin{cases} x \\ y \\ z \end{cases}.$

The whole *vis viva* is therefore

$$\omega'^2\ \Sigma\delta m\ (y^2 + z^2) + \omega''^2\ \Sigma\delta m\ (z^2 + x^2) + \omega'''^2\ \Sigma\delta m\ (x^2 + y^2).$$

Now the whole *vis viva* being constant, it can make no difference about what axes we estimate it. Hence we have

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = h \ \ldots . \ (1).$$

Again, since there are no forces acting, the sum of the mass of each element into the projection of the area described in an unit of time on any one of the co-ordinate planes, is constant with reference to the time. Or, considering the plane yz,

$$\Sigma\delta m \left(z\,\frac{dy}{dt} - y\,\frac{dz}{dt} \right) = \text{constant.}$$

Now the projections on the planes z_1y_1, y_1x_1 x_1z_1, are

$$\left.\begin{array}{l} \Sigma\delta m \left(z_1\,\dfrac{dy_1}{dt} - y_1\,\dfrac{dz_1}{dt} \right) \\[2ex] \Sigma\delta m \left(y_1\,\dfrac{dx_1}{dt} - x_1\,\dfrac{dy_1}{dt} \right) \\[2ex] \Sigma\delta m \left(x_1\,\dfrac{dz_1}{dt} - z_1\,\dfrac{dx_1}{dt} \right) \end{array}\right\} \text{ or } \begin{array}{c} A\omega_1 \\[2ex] B\omega_2 \\[2ex] C\omega_3 \end{array}$$

Hence, by the usual method in Geometry, the projection on the plane yz will be the sum of these projections, each multiplied into the cosines of the angles between the planes; or we have

$$\text{similarly}\quad \left.\begin{array}{l} Aa_1\omega + Bb\omega_2 + Cc_3\omega = k_1 \\ Aa'\omega_1 + Bb'\omega_2 + Cc'\omega_3 = k_2 \\ Aa''\omega_1 + Bb''\omega_2 + Cc''\omega_3 = k_3 \end{array}\right\} (a).$$

Adding their squares, we have

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = k^2 \ \ldots . \ (2).$$

Again, from the first of equations (a),

$$A\left(a\,\frac{d\omega_1}{dt} + \omega_1\,\frac{da}{dt}\right) + B\left(b\,\frac{d\omega_2}{dt} + \omega_2\,\frac{db}{dt}\right) + C\left(c\,\frac{d\omega_3}{dt} + \omega_3\,\frac{dc}{dt}\right) = 0.$$

Now it may easily be proved (Poisson, Art. 411,) that

$$\left.\begin{array}{l} \dfrac{dc}{dt} = \omega_2 a - \omega_1 b \\[2ex] \dfrac{db}{dt} = \omega_1 c - \omega_3 a \\[2ex] \dfrac{da}{dt} = \omega_3 b - \omega_2 c \end{array}\right\}.$$

Substituting these in the above equation, it becomes

$$a\left\{A\frac{d\omega_1}{dt}+(C-B)\,\omega_2\,\omega_3\right\}+b\left\{B\frac{d\omega_2}{dt}+(A-C)\,\omega_1\omega_3\right\}$$
$$+c\left\{C\frac{d\omega_3}{dt}+(B-A)\,\omega_1\omega_2\right\}=0.$$

We shall obtain similar equations with $a'b'c'$ and $a''b''c''$ in place of abc. Multiplying these by $aa'a''$, and taking notice of the equations of condition,

$$ab+a'b'+a''b''=0,\quad ac+a'c'+a''c''=0,\quad \text{and } a^2+a'^2+a''^2=1,$$

$$\text{we have } A\frac{d\omega_1}{dt}+(C-B)\,\omega_2\omega_3=0\ldots.(3).$$

The equations (1), (2), (3), completely determine the motion, and eliminating $\omega_2\omega_3$, we have for finding w_1 in terms of t the equation

$$t=\int\frac{\pm A\sqrt{(BC)}\,d\omega_1}{\sqrt{\{A(A-C)\,\omega_1^2-(k^2-Ch)}\ \sqrt{(k^2-Bh-A(A-B)\,\omega_1^2}},$$

which may be reduced to an elliptic function; so that

$$t=\pm\sqrt{\left\{\frac{ABC}{(A-C)(k^2-Bh)}\right\}}\int\frac{d\phi}{\sqrt{(1-c^2\sin^2\phi)}},$$

$$\text{where } c^2=\frac{(Ah-k^2)}{(k^2-Bh)}\cdot\frac{C-B}{A-C},$$

$$\text{and } \omega_1^2=\frac{k^2-Ch}{A(A-C)}\cdot\frac{1}{1-c^2\sin^2\phi}.$$

Similarly ω_2, ω_3, may be found in terms of t, and thence the position of the body at any time.

$\alpha.\ \beta.\ \gamma.$

X.—MATHEMATICAL NOTES.

1. *Demonstration of the principle of virtual velocities.*—If X, Y, Z, be the resolved parts of the moving forces which act on a particle m, the necessary and sufficient conditions of equilibrium are

$$X=0,\quad Y=0,\quad Z=0.$$

If a system of particles act on each other, the resolved parts of the force arising from the other particles which acts on each particle of the system, may be represented by the differential coefficients, taken with regard to the co-ordinates of the particle acted on, of a certain function; R of the mutual distances of the particles; and if X, Y, Z, represent the

resolved parts of all other forces acting on the particle, we have for each particle a set of equations of the form

$$X+\frac{dR}{dx}=0,\ Y+\frac{dR}{dy}=0,\ Z+\frac{dR}{dz}=0.$$

If we multiply each of such equations by infinitesimal arbitrary quantities δx, δy, δz, and add them together, we obtain

$$\Sigma\,(X\delta x+Y\delta y+Z\delta z)+\delta R=0.$$

If δx, δy, δz are proportional to any small possible displacements of the particle consistent with the preservation of the form of the system, $\delta R=0$; and we obtain finally

$$\Sigma\,(X\delta x+Y\delta y+Z\delta z)=0$$

as the equation which must be satisfied in all cases of equilibrium. This equation is the analytical expression of the principle of virtual velocities.

In order to deduce from this expression the six equations for the motion of a solid body, let $x+\delta x$, $y+\delta y$, $z+\delta z$, be the co-ordinates after displacement of a particle whose original co-ordinates are x, y, z; or, which is the same thing, be the co-ordinates of the particle referred to a system of axes which are slightly removed from the first, we have

$$\begin{aligned} x+\delta x &= \alpha + ax + by + cz,\\ y+\delta y &= \beta + a'x + b'y + c'z,\\ z+\delta z &= \gamma + a''x + b''y + c''z, \end{aligned}$$

where α, β, γ represent the co-ordinates of the displacement of the origin, a, b, c the cosines of the angles which the new axis of x makes with the original co-ordinate axes, and similarly for $a', b', c', a'', b'', c''$; so that we have the well known relations

$$\begin{aligned} aa'+bb'+cc' &= 0\\ aa''+bb''+cc'' &= 0\\ a'a''+b'b''+c'c'' &= 0. \end{aligned}$$

If the motions are infinitesimal, a, b', c'', differ from unity by infinitesimals of the second order, and the other quantities are infinitesimals of the first order; the equations are therefore reduced to

$$\begin{aligned} \delta x &= \alpha + by + cz, \qquad & a'+b &= 0,\\ \delta y &= \beta + a'x + c'z, & a''+c &= 0,\\ \delta z &= \gamma + a''x + b''y; & b''+c' &= 0. \end{aligned}$$

Eliminating a', b'', c'', we get

$$\delta x = a + by - a''z,$$
$$\delta y = \beta + c'z - bx,$$
$$\delta z = \gamma + a''x - a'y;$$

substituting these values in the equation

$$\Sigma\,(X\delta x + Y\delta y + Z\delta z) = 0, \text{ we obtain}$$

$$a\Sigma X + \beta\Sigma Y + \gamma\Sigma Z + b\Sigma\,(Xy - Yx) + c'\Sigma\,(Yz - Zy) + a''\Sigma\,(Zx - Xz) = 0,$$

and $a\beta\gamma$, $a''bc'$, being arbitrary, we obtain the common equations of equilibrium.

a. σ.

2. *Problem from the Papers of* 1842.—If $F(x, y, z) = \phi\,(u, v, w)$, where F is homogeneous of the n^{th} degree in x, y, z, and $u = \dfrac{dF}{dx}$, $v = \dfrac{dF}{dy}$, $w = \dfrac{dF}{dz}$; then

$$x = (n-1)\frac{d\phi}{du}, \quad y = (n-1)\frac{d\phi}{dv}, \quad z = (n-1)\frac{d\phi}{dw}.$$

Since F is homogeneous of n dimensions in x, y, z, we have

$$nF = x\frac{dF}{dx} + y\frac{dF}{dy} + z\frac{dF}{dz} = xu + yv + zw.$$

Hence $ndF = xdu + ydv + zdw + udx + vdy + wdz$,

or $(n-1)\,dF = xdu + ydv + zdw$.

But $(n-1)\,dF = (n-1)\,d\phi = (n-1)\left(\dfrac{d\phi}{du}\,du + \dfrac{d\phi}{dv}\,dv + \dfrac{d\phi}{dw}\,dw\right)$.

Therefore equating the coefficients of the differentials,

$$x = (n-1)\frac{d\phi}{du}, \quad y = (n-1)\frac{d\phi}{dv}, \quad z = (n-1)\frac{d\phi}{dw}.$$

ε.

3. The following mathematical expression for the *discontinuous* law of the sliding scale in the new Corn Bill, may be interesting to some of our readers.

Let p be the price of the quarter of corn in shillings, d the duty; then the formula expressing d in terms of p is

$$d = \frac{20}{1 + 0^{50-p}} + \frac{1}{1 + 0^{p-50} + 0^{74-p}}\left(72 - p - \frac{2}{1 + 0^{53-p}} + \frac{2}{1 + 0^{p-67}}\right).$$

In like manner the expression for the proposed Income Tax on a property x, is

$$t = \frac{7}{240}\,\frac{1}{1 + 0^{x-150}}\left(x - \frac{300}{1 + 0^{150-x}}\right).$$

a. σ.

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I.—ON A DIFFICULTY IN THE THEORY OF ALGEBRA.

By D. F. GREGORY, M.A. Fellow of Trinity College.

I OUGHT perhaps to apologize to the reader for calling his attention to a subject so much discussed as the nature of the symbols + and − when used in symbolical Algebra. But the theory which I wish to develop appears to me to remove some part of the difficulties which, after all which has been written, still adhere to the subject; and I am the more anxious to explain my views, because I have in previous papers held, in common I believe with every other writer, an opinion which a more attentive consideration induces me to think erroneous. It is generally assumed that the symbols + and − signify primarily addition and subtraction, and that any other meanings which we may attach to them must be derived from the fundamental significations. The theory which I have now to maintain is the apparently paradoxical one, that the symbols + and − do not represent the arithmetical operations of addition and subtraction; and that though they were originally intended to bear these meanings, they have become really the representatives of very different operations. This is opposed to our preconceived ideas, but I trust that the following statements will show the truth of the assertion.

When it is said that any symbol does not represent a particular operation, it is necessary to explain what is to be understood by an *Algebraical* symbol, and in what way it represents an operation. In previous papers on the Theory of Algebra, I have maintained the doctrine that a symbol is defined *algebraically* when its laws of combination are given; and that a symbol represents a given operation when the laws

of combination of the latter are the same as those of the former. This, or a similar theory of the nature of Algebra, seems to be generally entertained by those who have turned their attention to the subject: but without in any degree leaning on it, we may say that symbols are actually subject to certain laws of combination, though we do not suppose them to be so defined; and that a symbol representing any operation must be subject to the same laws of combination as the operation it represents. These assertions are independent of any theory except this, that there is a general Symbolical Algebra different from Arithmetical Algebra, in which the laws of the combination of the symbols are attended to. When therefore I say that the symbol + does not represent the arithmetical operation of addition, or − that of subtraction, I mean that the laws of combination of the symbols + and − are not those of the operations of addition and subtraction. Now the laws of combination of the symbols + and − are four, viz.

$$++a=+a, \quad +-a=-a, \quad -+a=-a, \quad --a=+a;$$

and whatever may be our theory of the meaning of these symbols, there is no doubt that we always assume them to be subject to these laws, which are indeed the very first steps which the student in algebra makes. But no one who considers the nature of the arithmetical operations can hesitate for a moment to say, that they are not subject to these laws; of which a sufficient proof is this, that addition and subtraction are inverse operations, whereas the second and third of the preceding laws are inconsistent with the idea that + and − are inverse symbols, the character of which is, that the one undoes what the other does; so that if f, ϕ are two symbols representing inverse operations, we have

$$f\phi(a) = a \quad \text{and} \quad \phi f(a) = a.$$

The same conclusion also follows from the fourth law, which is evidently analogous to the relation between a number and its square, not to that between a number and its reciprocal. A natural objection which may be brought against the view which I am here maintaining is, that we do actually define + and − to be the symbols representing addition and subtraction, and therefore that they must represent these operations. A further examination however will shew that this is not the case. We say that $a + x$ is to represent x added to a, and $a - x$, x subtracted from a: we do not directly assert that + signifies addition, and − subtraction. If we did we should contradict ourselves, when we asserted that $+-a=-a$, or $--a=+a$. The fact is, that we are deceived by writing

a *sum* and *difference* in a manner different from that in which we express the performance of any other operation. When we wish to denote that an operation f is performed on a subject a, we usually prefix the symbol of operation to the subject, and write $f(a)$: this however is not necessary, for we might connect them in any other way; and indeed Mr. Murphy prefixes the subject to the symbol of operation, apparently for the purpose of avoiding the prejudices which our ordinary mode of writing is apt to produce. But, on the other hand, when we wish to denote the addition of a number x to a number a, or its subtraction from it, we write

$$a+x \text{ or } a-x,$$

the operations being indicated by writing the symbol of the number added or subtracted *after* the subject, and separated from it by certain symbols; whereas in the ordinary mode of writing we should *prefix* the operating symbol. Now if we merely assert that $a+x$ is to signify the addition of x to a, the symbol $+$ might be understood to *indicate* addition, it being used to distinguish the kind of operation involving x which is performed on a, in the same way as $a \times x$ indicates the multiplication of a by x. But by such an assertion we do not make $+$ an *algebraical* symbol in the sense in which I use the word; nor does it *represent* the operation, though it may indicate it. It is only when we arrive at such conclusions as $a+(x+y)=a+x+y$, involving the law $++a=+a$, that we give to $+$ an algebraical individuality as a symbol subject to certain laws of combination, which, we see at once, are not those belonging to the operation of addition. If on this any one chooses to say that he considers $+$ as indicating (he cannot say *representing*) the operation of addition, and that he does not trouble himself about its laws of combination, there can be no objection to his holding such an opinion except this, that the Algebra in which such a symbol is used is not a general science, but simply Arithmetic. He cannot, consistently with this doctrine, hold that direction in Geometry can be indicated by $+$ and $-$; or that these symbols can receive any other interpretation than that which was originally assigned to them: conclusions inconsistent with any conceptions which we can form of a *general* Algebra. There is no doubt that we *can* give these symbols such a geometrical interpretation, and it is the possibility of so doing which has occasioned the difficulties of the Theory of Algebra considered as something more general than Arithmetic, and which has led to the more extended views which in recent years have been taken of the subject.

The preceding observations may be illustrated by using new symbols to represent the operations of addition and subtraction, by prefixing them in the ordinary way to the subject, and so investigating their laws of combination. Let us assume the symbol A to be that which represents addition, and B that of subtraction, and let us attach to them as a suffix the quantity which is added or subtracted; so that

$$A_x(a) \text{ and } B_x(a)$$

represent the addition of x to a and the subtraction of x from a. One of the most obvious laws of the operation of addition is, that if x and y be two quantities which are added to a, it is indifferent in what order the operations are performed: so that if we first add x to a, and then y to the sum, we obtain the same result as if we first added y to a, and then x to the sum. This law is expressed by the equation

$$A_x A_y(a) = A_y A_x(a);$$

that is, the operations A_x, A_y are commutative.

Again: another law is, that each of these sums is the same as if y were first added to x, and then that sum added to a. Now the sum of y and x is represented by $A_y(x)$, and therefore the addition of this sum to a is represented by $A_{A_y(x)}(a)$; so that the law in question is expressed by the equation

$$A_x A_y(a) = A_{A_y(x)}(a).$$

The same law may be extended to any number of variables x, y, z, so that we have the equation

$$A_x A_y A_z = A_{A_{A_z(y)(x)}}(a), \text{ and so on.}$$

The notation is an inconvenient one: but it is not here introduced for the purpose of supplanting the common one, but merely to show how the laws of combination of the operation of addition may be represented by one symbol only, without the aid of a subsidiary symbol such as $+$.

A third law of the operation of addition is, that it is indifferent whether x be added to a, or a to x: this is expressed by the equation

$$A_x(a) = A_a(x).$$

These three laws of combination are sufficient for our purpose, and show distinctly in a symbolical form how different the laws of the operation of addition are from the laws of combination of the symbol $+$, which therefore cannot represent it.

With regard to subtraction, since it is the operation which is inverse to addition, we have plainly

$$A_x B_x (a) = B_x A_x (a) = a.$$

Also, without going into detail, it is easy to see that

$$B_x A_y (a) = A_{B_x (y)}(a) = B_{B_y (x)}(a);$$

and thus the laws of the operation of subtraction are represented by means of the symbols of addition and subtraction.

By means of this notation we see distinctly how the symbol + appears frequently as a *separative* symbol between two others, in such a way that the order of the symbols cannot be changed. Thus we cannot say $+ax$ instead of $a+x$, though we do say that $a\sqrt{-1}\,x$, or $a(-)^{\frac{1}{2}}x$, is equivalent to $(-)^{\frac{1}{2}}ax$: for $a+x$ is equivalent to $A_x(a)$, whereas $a(-)^{\frac{1}{2}}x$ signifies only the successive performance of these operations which are commutative; the symbol $\sqrt{-1}$, or $(-)^{\frac{1}{2}}$, not having acquired the double signification which is attached in consequence of their position, to + or −. In the same way, though we say that $a+(-x) = a-x$, we do not say that $a+\sqrt{-1}\,x$, or $a+[(-)^{\frac{1}{2}}x] = a(-)^{\frac{1}{2}}x$, because these two formulæ are equivalent to $A_{-x}(a)$ and $A_{(-)^{\frac{1}{2}}x}(a)$, and the law that $A_{-x}(a) = B_x(a)$ has no analogue in the case of $(-)^{\frac{1}{2}}$, or other powers of + and −.

The distributive law, which is met with so frequently in algebraical operations, and which is usually written

$$f(a+x) = f(a) + f(x),$$

is in this notation expressed by the equation

$$f[A_x(a)] = A_{f(x)}[f(a)].$$

In like manner the index law, which is usually written

$$f^m f^n (a) = f^{m+n}(a),$$

becomes in this notation

$$f^m f^n (a) = f^{A_n(m)}(a).$$

To one who is acquainted with the higher branches of mathematics it is obvious that the operation, which is here denoted by A, is the same in kind as that of which so much use is made in the Calculus of Finite Differences for converting $f(x)$ into $f(x+h)$, and which has been represented by

the symbol D^k in the papers on the subject in preceding numbers of this journal.

Before concluding I would say a few words on what appears to me to be a prejudice relative to the nature of the symbols + and −. These are generally considered to be absolutely distinct from literal symbols, and have in consequence a different name assigned to them, being called "signs of affection." Such a distinction exists in arithmetical, but not in general Algebra. In the former, literal symbols are used to represent numbers or magnitudes, and are capable of receiving interpretation, that is, of having different meanings or values assigned to them, while the signs of affection indicate the performance of certain operations, and are incapable of bearing any other meaning than those which are originally assigned to them. As such, the signs + and − are exactly on a par with × and ÷, though the latter, from accidental circumstances, have not become so important as the former. When we write the symbol a in Arithmetical Algebra, we mean that we may substitute for it any number we choose; but when we write $a+b$, we say that b is added to a, we attach to + a definite meaning, and we can give no other interpretation to it, without taking into consideration its laws of combination, which are excluded from Arithmetical Algebra. On the other hand, in Symbolical Algebra, where every symbol represents an operation, it is obvious that we cannot *à priori* speak of any difference in kind between different symbols. In such a science what is a, and what is +? To neither are definite meanings attached, as to the latter symbol in Arithmetical Algebra. Our conceptions would be clearer, and our minds more free from prejudice, if we never used in the general science symbols to which definite meanings had been appropriated in the particular science. Inveterate practice has however so wedded us to the use of the symbols + and − that we find it difficult to dispense with them, and still more difficult, in using them, to avoid being misled by ideas drawn from Arithmetic. The symbols + and −, × and ÷, were invented for the purpose of indicating the performance of certain operations on numbers; but as the science advanced, it was found that the symbol × might be conveniently omitted, the operation being indicated merely by the juxtaposition of symbols; so that ax stood for $a \times x$. From this the transition was easy to the conception of a as the symbol of the operation; a change of great importance, as leading to the view that Symbolical Algebra is a Calculus of Operations. But it is merely a matter of accident that the symbol × was that which was expunged: that fate might as

well have befallen the symbol +, and then $a\,x$ would have signified the addition of a to x, and the difficulties which have been experienced regarding + and − would then have been transferred to × and ÷. It is perhaps, to a certain extent, unfortunate that we have in multiplication represented by one letter the symbol of the operation as well as that with respect to which it is performed: the latter ought rather to be attached as an index or a suffix. Thus, if we represented the multiplication of x by a by the symbol $P_a(x)$, we should have no difficulty in seeing that it was exactly analogous to addition under the notation $A_a(x)$, as I have in this paper written it.

In the preceding remarks I have proceeded on the supposition that Symbolical Algebra must be considered as a science of operations represented symbolically: this view may not appear to every one necessary; but if the subject be considered in all its generality, it will, I am convinced, be found that there is no other way of explaining the difficulties of Algebra in a uniform and consistent manner.

II.—ON THE LIMITS OF MACLAURIN'S THEOREM.

By A. Q. G. Craufurd, M.A. Jesus College.

To the Editor of the Cambridge Mathematical Journal.

Sir,—There is an error in my last paper which I did not observe till I saw it in print, and which I now hasten to correct. But before doing so, I must give the reader notice, that in what follows I shall use the symbol $\underset{a^n}{C}$ instead of $\overset{a}{\underset{n}{C}}$. The former is more analogous than the latter to the ordinary notation of algebra, and may be usefully extended by taking $\underset{e^{nx}}{C}$ to represent the coefficient of e^{na} in a series of powers of e^a; and $\underset{\cos na}{C}$ for the coefficient of cos na in a series of the cosines of multiples of a.

The error to which I have alluded occurs in page 85, line 10, where, having shewn that the terms of Maclaurin's series which follow that affected with x^n, may be represented by

$$x^{n+1}\underset{a^{n+1}}{C}\cdot\frac{f(a)}{1-\dfrac{x}{a}},\ \left(\text{or } x^{n+1}\underset{a^n}{C}\cdot\frac{f(a)}{a-x},\right)$$

I proceed to substitute for this expression

$$\frac{x^{n+1}}{1.2\ldots.(n+1)}\cdot\left\{\frac{d^{n+1}}{da^{n+1}}\cdot\frac{f(a)}{1-\dfrac{x}{a}}\right\}_{a=0},$$

which is the same as

$$\frac{x^{n+1}}{1.2\ldots.n}\cdot\left\{\frac{d^n}{da^n}\,\frac{f(a)}{a-x}\right\}_{a=0}.$$

I was led to this conclusion by observing, that the quantity $\dfrac{f(a)}{a-x}$ *may be* developed in a series of positive powers of a; and from this I inferred, that the coefficient of a^n in its development must be equal to $\dfrac{1}{1.2\ldots.n}\times$ (the value corresponding to $a=0$ of its n^{th} differential coefficient). But this argument is fallacious, for the expression $\underset{a^n}{C}.\dfrac{f(a)}{a-x}$ is indeterminate; it has at least two values, and

$$\frac{1}{1.2\ldots.n}\cdot\left\{\frac{d^n}{da^n}\cdot\frac{f(a)}{a-x}\right\}_{a=0}$$

is one of these values, but not the right one. Supposing $f(a)$ to be developable only in a series of positive powers of a, the values of $\underset{a^n}{C}.\dfrac{f(a)}{a-x}$ are

$$\underset{a^{n+1}}{C}f(a)+x\underset{a^{n+2}}{C}f(a)+x^2\underset{a^{n+3}}{C}f(a)+\&\text{c. to infinity},$$

and $-\{x^{-1}\underset{a^n}{C}f(a)+x^{-2}\underset{a^{n-1}}{C}f(a)+x^{-3}\underset{a^{n-2}}{C}f(a)\ldots.x^{-(n+1)}\underset{a^0}{C}f(a)\}$.

The former is that which results from developing $\dfrac{1}{a-x}$ in a series of negative powers of a; the latter results from developing the same in a series of positive powers. If each of these series be multiplied by x^{n+1}, the former gives those terms of Maclaurin's series which follow that affected with x^n (or *the remainder*); the latter, which is equal to

$$\frac{1}{1.2\ldots.n}\cdot\left\{\frac{d^n}{da^n}\cdot\frac{f(a)}{a-x}\right\}_{a=0},$$

gives all the terms up to that affected with x^n, with their signs changed, which are equivalent to the remainder $-f(x)$.

Consequently, the only real results of my paper in the 14th number of the Journal, are those which give the sum of any

part of the series by means of the symbol $\underset{a^n}{C}$, and those which give the sums of a finite number of terms by means of the symbol of differentiation; the latter may be written thus.

The terms of Maclaurin's series from that affected with x^n to that affected with x^m, both included, are equivalent to

$$\frac{x^n}{1.2\ldots\ldots m}\left\{\frac{d^m}{da^m}\cdot f(a)\cdot\frac{x^{m-n+1}-a^{m-n+1}}{x-a}\right\}_{a=0}.$$

When $n=0$, this becomes

$$\frac{1}{1.2.3\ldots\ldots m}\left\{\frac{d^m}{da^m}\cdot f(a)\cdot\frac{x^{m+1}-a^{m+1}}{x-a}\right\}_{a=0},$$

$$=\frac{x^{m+1}}{1.2.3\ldots\ldots m}\left\{\frac{d^m}{da^m}\cdot\frac{f(a)}{x-a}\right\}_{a=0};$$

which is the value of all the terms up to that affected with x^m

The corresponding expressions for Taylor's series are,

$$\frac{h^n}{1.2\ldots\ldots m}\left\{\frac{d^m}{da^m}\cdot f(x+a)\cdot\frac{h^{m-n+1}-a^{m-n+1}}{h-a}\right\}_{a=0},$$

and $$\frac{1}{1.2\ldots\ldots m}\left\{\frac{d^m}{da^m}\cdot f(x+a)\cdot\frac{h^{m+1}-a^{m+1}}{h-a}\right\}_{a=0},$$

$$=\frac{h^{m+1}}{1.2.3.\ldots\ldots m}\left\{\frac{d^m}{da^m}\cdot\frac{f(x+a)}{h-a}\right\}_{a=0}.$$

Cor. The portions of the two series intermediate between the terms affected with the m^{th} and n^{th} powers (but including them), may also be represented by

$$\frac{x^{m+1}}{1.2\ldots m}\left\{\frac{d^m}{da^m}\cdot\frac{f(a)}{x-a}\right\}_{a=0}-\frac{x^n}{1.2\ldots(n-1)}\left\{\frac{d^{n-1}}{da^{n-1}}\cdot\frac{f(a)}{x-a}\right\}_{a=0},$$

and

$$\frac{h^{m+1}}{1.2\ldots m}\left\{\frac{d^m}{da^m}\cdot\frac{f(x+a)}{h-a}\right\}_{a=0}-\frac{h^n}{1.2\ldots(n-1)}\left\{\frac{d^{n-1}}{da^{n-1}}\cdot\frac{f(x+a)}{h-a}\right\}_{a=0};$$

which expressions are easily reduced to those previously given.

London, April 6th, 1842.

III.—ON CERTAIN EXPANSIONS, IN SERIES OF MULTIPLE SINES AND COSINES.

By Arthur Cayley, B.A. Trin. Coll.

In the following paper we shall suppose ε the base of the hyperbolic system of logarithms; e a constant, such that its modulus, and also the modulus of $\frac{1}{e}\{1-\sqrt{(1-e^2)}\}$, are each of them less than unity; $\chi\{\varepsilon^{u\sqrt{(-1)}}\}$ a function of u, which, as (u) increases from 0 to π, passes continuously from the former of these values to the latter, without becoming a maximum in the interval, $f(\varepsilon^{u\sqrt{(-1)}}$ any function of (u) which remains finite and continuous for values of u included between the above limits. Hence, writing

$$\chi\{\varepsilon^{u\sqrt{(-1)}}\}=m \;\ldots\ldots\ldots\; (1),$$

and considering the quantity

$$\frac{\sqrt{(1-e^2)}\,.\,f\{\varepsilon^{u\sqrt{(-1)}}\}}{\sqrt{(-1)}\,\chi'\{\varepsilon^{u\sqrt{(-1)}}\}\,(1-e\cos u)}\;\ldots\; (2),$$

as a function of m, for values of m or u included between the limits 0 and π, we have

$$\frac{\sqrt{(1-e^2)}\,.\,f\{\varepsilon^{u\sqrt{(-1)}}\}}{\sqrt{(-1)}\,\chi'\{\varepsilon^{u\sqrt{(-1)}}\}(1-e\cos u)}$$

$$=\frac{2}{\pi}\,\Sigma_{-\infty}^{\infty}\cos rm\int_0^\pi\frac{\sqrt{(1-e^2)}\,.\,f\{\varepsilon^{u\sqrt{(-1)}}\}\cos rm\,dm}{\sqrt{(-1)}\,\chi'\{\varepsilon^{u\sqrt{(-1)}}\}\,(1-e\cos u)}\;\ldots\; (3),$$

(Poisson, *Mec.* tom. I. p. 650); which may also be written

$$\frac{\sqrt{(1-e^2)}\,.\,f\{\varepsilon^{u\sqrt{(-1)}}\}}{\sqrt{(-1)}\,\chi'\{\varepsilon^{u\sqrt{(-1)}}\}\,(1-e\cos u)}$$

$$=\frac{2}{\pi}\,\Sigma_{-\infty}^{\infty}\cos rm\int_0^\pi\frac{\sqrt{(1-e^2)}\,.\,f\{\varepsilon^{u\sqrt{(-1)}}\}\cos r\chi\{\varepsilon^{u\sqrt{(-1)}}\}\,du}{1-e\cos u}\;\ldots\; (4).$$

And if the first side of the equation be generally expansible in a series of multiple cosines of m, instead of being so in particular cases only, its expanded value will always be the one given by the second side of the preceding equation.

Now, between the limits 0 and π, the function

$$f\{\varepsilon^{u\sqrt{(-1)}}\}\cos r\chi\{\varepsilon^{u\sqrt{(-1)}}\}$$

will always be expansible in a series of multiple cosines of u; and if by any algebraical process the function $f\rho\cos r\chi\rho$ can be expanded in the form

$$f\rho\cos r\chi\rho \doteq \Sigma_{-\infty}^{\infty}a_s\rho^s,\quad (a_s=a_{-s})\;\ldots\ldots\; (5);$$

we have, in a convergent series,

$$f\{\varepsilon^{u\sqrt{(-1)}}\}\cos r\chi\,\{\varepsilon^{u\sqrt{(-1)}}\} = a_0 + 2\Sigma_1^\infty\, a_s \cos su \,\ldots\,(6).$$

Again, putting $\dfrac{1}{e}\{1 - \sqrt{(1-e^2)}\} = \lambda$ ....(7),

we have $$\frac{\sqrt{(1-e^2)}}{1 - e\cos u} = 1 + 2\Sigma_1^\infty\,\lambda^p \cos pu\,\ldots\ldots\,(8).$$

Multiplying these two series, and effecting the integration, we obtain

$$\frac{1}{\pi}\int_0^\pi \frac{\sqrt{(1-e^2)}\,.\,f\{\varepsilon^{u\sqrt{(-1)}}\}\cos r\chi\,\{\varepsilon^{u\sqrt{(-1)}}\}\,du}{1 - e\cos u} = 2\{\tfrac{1}{2}a_0 + \Sigma_1^\infty(a_s\lambda^s)\}\ldots(9).$$

And the second side of this equation being obviously derived from the expansion of $f\lambda\cos r\chi\lambda$, by rejecting negative powers of λ, and dividing by 2 the term independent of λ, may conveniently be represented by the notation

$$\underbrace{2f\lambda\cos r\chi\lambda}\ \ldots\ldots\ldots\ (10);$$

where in general, if $\Gamma\,.\,\lambda$ can be expanded in the form

$$\Gamma\,.\,\lambda = \Sigma_{-\infty}^\infty\,(A_s\lambda^s), \quad [A_{-s} = A_s]\,\ldots(11),$$

we have $$\underbrace{\Gamma\,.\,\lambda} = \tfrac{1}{2}A_0 + \Sigma_1^\infty\, A_s\lambda^s\,\ldots\ldots\,(12).$$

(By what has preceded, the expansion of $\Gamma\,.\,\lambda$ in the above form is always possible in a certain sense; however, in the remainder of the present paper, $\Gamma\,.\,\lambda$ will always be of a form to satisfy the equation $\Gamma\,.\left(\dfrac{1}{\lambda}\right) = \Gamma\,.\,\lambda$, except in cases which will afterwards be considered, where the condition $A_{-s} = A_s$ is unnecessary.)

Hence, observing the equations (4), (9), (10),

$$\frac{\sqrt{(1-e^2)}\,f\varepsilon^{u\sqrt{(-1)}}}{\sqrt{(-1)}\,\chi'\{\varepsilon^{u\sqrt{(-1)}}\}\,(1 - e\cos u)} = \Sigma_{-\infty}^\infty\,\cos rm\,\underbrace{2\cos r\chi\lambda f\lambda}\ldots(13);$$

from which, assuming a system of equations analogous to (1), and representing by $\Pi(\Phi)$ the product $\Phi_1\Phi_2\ldots$, it is easy to deduce

$$\Pi\left\{\frac{\sqrt{(1-e^2)}}{\sqrt{(-1)}\,\chi'\{\varepsilon^{u\sqrt{(-1)}}\}\,(1 - e\cos u)}\right\}.\,f\{\varepsilon^{u_1\sqrt{(-1)}},\ \varepsilon^{u_2\sqrt{(-1)}}\ldots\}$$
$$= \Sigma_{-\infty}^\infty\,\Sigma_{-\infty}^\infty\ldots\Pi\cos rm\,\underbrace{\Pi(2\cos r\chi\lambda)\,f(\lambda_1,\lambda_2\ldots)}\ldots(14),$$

where $\Gamma\,.\,(\lambda_1, \lambda_2\ldots)$ being expansible in the form

$$\Gamma.(\lambda_1,\lambda_2\ldots) = \Sigma_{-\infty}^\infty\,\Sigma_{-\infty}^\infty\ldots A_{s_1,\,s_2\ldots}\lambda_1^{s_1}\lambda_2^{s_2}\ldots \quad [A_{s_1,\,s_2\ldots} = A_{-s_1,\,-s_2\ldots}]$$
$$\ldots\ (15),$$

$$\Gamma.(\lambda_1, \lambda_2 \ldots) = \Sigma_0^\infty\, \Sigma_0^\infty \ldots \frac{1}{2^N} A_{s_1, s_2 \ldots}\, \lambda_1^{s_1} \lambda_2^{s_2} \ldots\ldots\ldots\ldots \text{(16)},$$

N being the number of exponents which vanish.

The equations (13) and (14) may also be written in the forms

$$f\{\varepsilon^{u\sqrt{(-1)}}\} = \Sigma_{-\infty}^{\infty} \cos rm\; \overbrace{\underbrace{2 \cos r\chi\lambda\, \frac{\sqrt{(-1)}\chi'\lambda\,\{1-\tfrac{1}{2}e(\lambda+\lambda^{-1})\}}{\sqrt{(1-e^2)}}}}\, f\lambda \ldots \text{(17)},$$

$$f\{\varepsilon^{u_1\sqrt{(-1)}}, \varepsilon^{u_2\sqrt{(-1)}} \ldots\}$$

$$= \Sigma_{-\infty}^{\infty}\, \Sigma_{-\infty}^{\infty} \ldots \Pi(\cos rm)\; \overbrace{\underbrace{\Pi\left\{2 \cos r\chi\lambda.\, \frac{\sqrt{(-1)}\chi'\lambda\{1-\tfrac{1}{2}e(\lambda+\lambda^{-1})\}}{\sqrt{(1-e^2)}}\right\} f(\lambda_1, \lambda_2 \ldots)}} \ldots \text{(18)}.$$

As examples of these formulæ, we may assume

$$\chi\{\varepsilon^{u\sqrt{(-1)}}\} = m = u - e \sin u \ldots\ldots \text{(19)}.$$

Hence, putting

$$\lambda^r \varepsilon^{-\frac{re}{2}(\lambda-\lambda^{-1})} + \lambda^{-r} \varepsilon^{\frac{re}{2}(\lambda-\lambda^{-1})} = \Lambda_r \ldots\ldots \text{(20)},$$

and observing the equation

$$\sqrt{(-1)}\,\chi'\{\varepsilon^{u\sqrt{(-1)}}\} = 1 - e \cos u \ldots\ldots \text{(21)},$$

the equation (17) becomes

$$f\{\varepsilon^{u\sqrt{(-1)}}\} = \Sigma_{-\infty}^{\infty} \cos rm\; \Lambda_r\, \overbrace{\underbrace{\frac{\{1-\tfrac{1}{2}e(\lambda+\lambda^{-1})\}^2}{\sqrt{(1-e^2)}}}} \ldots \text{(22)}.$$

Thus, if

$$\theta - \pi = \cos^{-1} \frac{\cos u - e}{1 - e \cos u} \ldots\ldots \text{(23)},$$

assuming

$$f\{\varepsilon^{u\sqrt{(-1)}}\} = \frac{\cos u - e}{1 - e \cos u} \ldots\ldots \text{(24)},$$

$$\cos(\theta - \pi) = \Sigma_{-\infty}^{\infty} \frac{1}{\sqrt{(1-e^2)}} \cos rm\; \overbrace{\underbrace{\left\{1 - \frac{e}{2}(\lambda+\lambda^{-1})\right\}\left\{\frac{1}{2}(\lambda+\lambda^{-1}) - e\right\}\Lambda_r}} \ldots \text{(25)},$$

the term corresponding to $r = 0$ being

$$\frac{1}{2\sqrt{(1-e^2)}}\{2\lambda - 2e - e(\lambda^2+1) + 2e^2\lambda\}, = -e \ldots \text{(26)}.$$

Again, assuming

$$f\{\varepsilon^{u\sqrt{(-1)}}\} = \frac{d\theta}{dm} = \frac{\sqrt{(1-e^2)}}{(1 - e \cos u)^2} \ldots\ldots \text{(27)},$$

and integrating the resulting equation with respect to m,

$$\theta - \pi = \Sigma_{-\infty}^{\infty} \frac{\sin rm}{r} \overbrace{\underbrace{\Lambda_r}} = m + 2\Sigma_1^\infty \frac{\sin rm}{r} \overbrace{\underbrace{\Lambda_r}} \ldots \text{(28)},$$

a formula given in the fifth Number of the *Mathematical Journal*, and which suggested the present paper.

As another example, let

$$f\,\varepsilon^{u\sqrt{(-1)}} = \cos(\theta - \pi)\,\frac{d\theta}{dm} = \frac{\sqrt{(1-e^2)}\,.\,(\cos u - e)}{(1 - e\cos u)^3} \;\ldots\ldots\; (29).$$

Then integrating with respect to m, there is a term

$$2m\,.\,\frac{\frac{1}{2}(\lambda + \lambda^{-1}) - e}{1 - \frac{1}{2}e(\lambda + \lambda^{-1})} \;\ldots\ldots\ldots\; (30),$$

which it is evident, *à priori*, must vanish. Equating it to zero, and reducing, we obtain

$$\frac{e}{1-e^2} = \frac{\lambda + \lambda^{-1}}{1 - \frac{e}{2}(\lambda + \lambda^{-1})} \;\ldots\ldots\ldots\; (31),$$

i.e. $$\frac{e}{1-e^2} = \lambda + \frac{e}{2}(\lambda^2 + 1) + \frac{e^2}{4}(\lambda^3 + 3\lambda) + \frac{e^3}{8}(\lambda^4 + 4\lambda^2 + 3) + \ldots\ldots (32),$$

a singular formula, which may be verified by substituting for λ its value: we then obtain

$$\sin(\theta - \pi) = 2\Sigma_1^\infty \frac{\sin rm}{r}\,\Lambda_r\,\frac{\frac{1}{2}(\lambda + \lambda^{-1}) - e}{1 - \frac{1}{2}(\lambda + \lambda^{-1})} \;\ldots\ldots\; (33).$$

The expansions of $\sin k(\theta - \pi)$, $\cos k(\theta - \pi)$, are in like manner given by the formulæ

$$\cos k(\theta - \pi) = \Sigma_{-\infty}^{\infty}\,\Lambda_r L \cos kL\,\cos rm \;\ldots\ldots\; (34),$$

$$\sin k(\theta - \pi) = \Sigma_{-\infty}^{\infty}\,\Lambda_r \frac{1}{kr}\cos kL\,\frac{\sin rm}{r} \;\ldots\ldots\; (35),$$

where, to abbreviate, we have written

$$\cos^{-1}\left\{\frac{\frac{1}{2}(\lambda + \lambda^{-1}) - e}{1 - \frac{e}{2}(\lambda + \lambda^{-1})}\right\} = L \;\ldots\ldots\ldots\; (36),$$

$$\frac{\left\{1 - \frac{e}{2}(\lambda + \lambda^{-1})\right\}^2}{\sqrt{(1-e^2)}} = L \;\ldots\ldots\ldots\; (37).$$

Forming the analogous expressions for

$$\cos k(\theta' - \pi'), \quad \sin k(\theta' - \pi'),$$

substituting in

$\cos k(\theta - \theta')$

$= \cos k(\pi - \pi')\,\{\cos k(\theta - \pi)\cos k(\theta' - \pi') + \sin k(\theta - \pi)\sin k(\theta' - \pi')\}$

$- \sin k(\pi - \pi')\,\{\sin k(\theta - \pi)\cos k(\theta' - \pi') - \sin k(\theta' - \pi')\cos k(\theta - \pi)\},$

and reducing the whole to multiple cosines, the final result takes the very simple form

$$\cos k\,(\theta-\theta')$$
$$=\Sigma_{-\infty}^{\infty}\cos\{r'm'-rm+k(\pi-\pi')\}\,\overbrace{\Lambda_r\Lambda'_{r'}\cos kL\cos kL'\left(L-\frac{1}{kr}\right)\left(L'-\frac{1}{kr'}\right)}\ldots(38).$$

Again, formulæ analogous to (14), (18), may be deduced from the equation

$$\Gamma.(m_1,\,m_2\ldots)$$
$$=\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots\begin{cases}\cos(r_1m_1+r_2m_2\ldots)\int_0^{2\pi}\frac{dm_1}{2\pi}\int_0^{2\pi}\frac{dm_2}{2\pi}\ldots\cos(r_1m_1+r_2m_2\ldots)\,\Gamma.(m_1,\,m_2\ldots)+\\ \sin(r_1m_1+r_2m_2\ldots)\int_0^{2\pi}\frac{dm_1}{2\pi}\int_0^{2\pi}\frac{dm_2}{2\pi}\ldots\sin(r_1m_1+r_2m_2\ldots)\,\Gamma.(m_1,\,m_2\ldots)\end{cases}\quad\ldots.\ (39),$$

which holds from $m_1=0$ to $m_1=2\pi$, &c., but in many cases universally. In this case, writing for $\Gamma.(m_1,\,m_2\ldots)$ the function

$$\Pi\left\{\frac{1}{\sqrt{(-1)}\,\chi'\{\varepsilon^{u\sqrt{(-1)}}\}}\;\frac{\sqrt{(1-e^2)}-e\sin u\sqrt{(-1)}}{1-e\cos u}\right\}f\{\varepsilon^{u_1\sqrt{(-1)}},\,\varepsilon^{u_2\sqrt{(-1)}}\ldots\}\ldots(40);$$

and observing

$$\frac{\sqrt{(1-e^2)}-e\sin u\sqrt{(-1)}}{1-e\cos u}=\frac{1+\lambda\varepsilon^{-u\sqrt{(-1)}}}{1-\lambda\varepsilon^{-u\sqrt{(-1)}}}$$
$$=1+2\Sigma_1^{\infty}\{\cos su-\sqrt{(-1)}\sin su\}\,\lambda^s\ \ldots.\ (41),$$

an exactly similar analysis, (except that in the expansion $\Gamma.(\lambda_1,\,\lambda_2\ldots)=\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots A_{s_1,s_2}\ldots\lambda_1^{s_1}\lambda_2^{s_2}\ldots$, the supposition is not made that $A_{s_1,\,s_2}\ldots=A_{-s_1,\,-s_2}\ldots$), leads to the result

$$f\{\varepsilon^{u_1\sqrt{(-1)}},\,\varepsilon^{u_2\sqrt{(-1)}}\ldots)\,\Pi\left\{\frac{\sqrt{(1-e^2)}-e\sin u\sqrt{(-1)}}{\sqrt{(-1)}\,\chi'\varepsilon^{u\sqrt{(-1)}}(1-e\cos u)}\right\}=$$
$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots\begin{cases}\cos(r_1m_1+r_2m_2\ldots)\,\overbrace{2^n\cos(r_1\chi_1\lambda_1+r_2\chi_2\lambda_2\ldots)f(\lambda_1,\lambda_2\ldots)}+\\ \sin(r_1m_1+r_2m_2\ldots)\,\underbrace{2^n\sin(r_1\chi_1\lambda_1+r_2\chi_2\lambda_2\ldots)f(\lambda_1,\lambda_2\ldots)}\end{cases}\quad\ldots.\ (42).$$

(n) being the number of variables $u_1,\,u_2\ldots$ This may also be written

$$f\{\varepsilon^{u_1\sqrt{(-1)}},\,\varepsilon^{u_2\sqrt{(-1)}}\ldots\}=$$
$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots\begin{cases}\cos(r_1m_1+\ldots)\,\overbrace{\cos(r_1\chi_1\lambda_1+\ldots)\,\Pi\left\{\frac{2\sqrt{(-1)}\chi'\lambda\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}}{\sqrt{(1-e^2)}-\frac{1}{2}e(\lambda-\lambda^{-1})}\right\}f(\lambda_1,\lambda_2\ldots)}+\\ \sin(r_1m_1+\ldots)\,\overbrace{\sin(r_1\chi_1\lambda_1+\ldots)\,\Pi\left\{\frac{2\sqrt{(-1)}\chi'\lambda\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}}{\sqrt{(1-e^2)}-\frac{1}{2}e(\lambda-\lambda^{-1})}\right\}f(\lambda_1,\lambda_2\ldots)}\end{cases}\quad\ldots.\ (43).$$

By choosing for $f\{\varepsilon^{u_1\sqrt{(-1)}}, \varepsilon^{u_2\sqrt{(-1)}}\ldots\}$, functions expansible without sines, or without cosines, a variety of formulæ may be obtained: we may instance

$$\overbrace{\underbrace{\frac{(\lambda-\lambda^{-1})\times\left\{1-\frac{e}{2}(\lambda+\lambda^{-1})\right\}\Lambda_r}{\sqrt{(1-e^2)}-\frac{e}{2}(\lambda-\lambda^{-1})}}} = 0 \;\ldots\ldots (44),$$

Λ_r having the same meaning as before.

Also,
$$\overbrace{\underbrace{\frac{\left\{\frac{1}{2}(\lambda+\lambda^{-1})-e\right\}\left\{1-\frac{e}{2}(\lambda+\lambda^{-1})\right\}\Lambda'_r}{\sqrt{(1-e^2)}-\frac{e}{2}(\lambda-\lambda^{-1})}}} = 0 \;\ldots\ldots (45),$$

where
$$\Lambda'_r = \lambda^r\,\varepsilon^{-\frac{re}{2}(\lambda-\lambda^{-1})} - \lambda^{-r}\,\varepsilon^{\frac{re}{2}(\lambda-\lambda^{-1})} \;\ldots\ldots\ldots\ldots (46).$$

Again,
$$\overbrace{\underbrace{\frac{\left\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\right\}(\lambda-\lambda^{-1})\Lambda'_r}{1-\frac{1}{2}\frac{e}{\sqrt{(1-e^2)}}(\lambda-\lambda^{-1})}}} + \frac{2}{r}\overbrace{\underbrace{\Lambda_r}} = 0 \;\ldots\ldots (47),$$

and

$$\overbrace{\underbrace{\frac{\left\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\right\}\left\{(\lambda+\lambda^{-1})-\frac{1}{2}e\right\}\Lambda_r}{1-\frac{1}{2}\frac{e}{\sqrt{(1-e^2)}}(\lambda-\lambda^{-1})}}} = \overbrace{\underbrace{\left\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\right\}\left\{(\lambda+\lambda^{-1})-\frac{1}{2}e\right\}\Lambda_r}}, \;\ldots\ldots (48);$$

or, what is the same thing,

$$\overbrace{\underbrace{\frac{(\lambda-\lambda^{-1})\left\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\right\}\left\{(\lambda+\lambda^{-1})-\frac{1}{2}e\right\}\Lambda_r}{1-\frac{1}{2}\frac{e}{\sqrt{(1-e^2)}}(\lambda-\lambda^{-1})}}} = 0 \ldots\ldots (49);$$

or, comparing with (44),

$$\overbrace{\underbrace{\frac{(\lambda^2-\lambda^{-2})\left\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\right\}\Lambda_r}{1-\frac{1}{2}\frac{e}{\sqrt{(1-e^2)}}(\lambda-\lambda^{-1})}}} = 0 \;\ldots\ldots\ldots (50),$$

which are all obtained by applying the formula (43) to the expansion of $\genfrac{}{}{0pt}{}{\sin}{\cos}(\theta-\pi)$, and comparing with the equations (25), (33).

IV.—ON SOME DEFINITE INTEGRALS.*

THE following paper contains examples of the Evaluation of Definite Integrals by means of artifices depending on the following equation, the truth of which it is easy to see, viz.

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \ \ldots\ldots\ (A).$$

1. To find the value of $\int_0^{\frac{1}{2}\pi} \log \sin\theta\, d\theta$.

Let $$u = \int_0^{\frac{1}{2}\pi} \log \sin\theta\, d\theta,$$

then $$u = \int_0^{\frac{1}{2}\pi} \log \cos\theta\,.\,d\theta, \text{ by (A)};$$

adding these, $$2u = \int_0^{\frac{1}{2}\pi} \log \sin\theta \cos\theta\,.\,d\theta$$
$$= \int_0^{\frac{1}{2}\pi} (\log \sin 2\theta + \log \tfrac{1}{2})\, d\theta;$$

but $$\int_0^{\frac{1}{2}\pi} \log \sin 2\theta\, d\theta = \tfrac{1}{2}\int_0^{\pi} \log \sin\theta'.d\theta' \text{ (by putting } 2\theta=\theta')$$
$$= \int_0^{\frac{1}{2}\pi} \log \sin\theta'\, d\theta',$$

(since the values of the sine between 0 and $\frac{1}{2}\pi$ exactly correspond to those between $\frac{1}{2}\pi$ and π,)

$$= u;$$

therefore $$2u = u + \log\tfrac{1}{2}\int_0^{\frac{1}{2}\pi} d\theta,$$
$$u = \tfrac{1}{2}\pi \log \tfrac{1}{2}.$$

2. Hence we can find $\int_0^{\pi} \theta \log \sin\theta\, d\theta$.

For $$\int_0^{\pi} \theta^2 \log \sin\theta\, d\theta = \int_0^{\pi} (\pi-\theta)^2 \log\sin\theta\, d\theta;$$

therefore $$0 = \int_0^{\pi} (\pi^2 - 2\pi\theta) \log \sin\theta\, d\theta,$$

$$\int_0^{\pi} \theta \log \sin\theta\, d\theta = \tfrac{1}{2}\pi \int_0^{\pi} \log \sin\theta\, d\theta = \tfrac{1}{2}\pi^2 \log \tfrac{1}{2}.$$

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* From a Correspondent.

3. In like manner,

$$\int_0^\pi \theta \sin^n \theta \, d\theta = \int_0^\pi (\pi - \theta) \sin^n \theta \, d\theta$$

$$= \tfrac{1}{2}\pi \int_0^\pi \sin^n \theta \, d\theta,$$

which is a known integral.

4. To find the value of $\int_0^\pi \frac{x \sin x \, dx}{1 + \cos^2 x}$.

$$\int_0^\pi \frac{x \sin x \, dx}{1 + \cos^2 x} = \int_0^\pi \frac{(\pi - x) \sin x \, dx}{1 + \cos^2 x}$$

$$= \tfrac{1}{2}\pi \int_0^\pi \frac{\sin x \, dx}{1 + \cos^2 x}$$

$$= \tfrac{1}{2}\pi \left\{\tan^{-1} 1 - \tan^{-1}(-1)\right\} = \tfrac{1}{4}\pi^2.$$

5. Suppose $f(a) = \int_0^\pi dx \log(1 - 2a \cos x + a^2)$.

Then $$f(a) = \int_0^\pi dx \log(1 + 2a \cos x + a^2);$$

adding these, it is easily seen that

$$2f(a) = \int_0^\pi dx \log(1 - 2a^2 \cos 2x + a^4)$$

$$= \tfrac{1}{2} \int_0^{2\pi} dx \log(1 - 2a^2 \cos x + a^4)$$

(by writing x instead of $2x$)

$$= \int_0^\pi dx \log(1 - 2a^2 \cos x + a^4) = f(a^2);$$

in like manner $2f(a^2) = f(a^4)$, and so on: but if a be less than 1, and n very large,

$$f(a^n) = \int_0^\pi dx \log(1) = 0,$$

therefore $f(a) = 0$;

if a be greater than 1,

$$f(a) = \int_0^\pi dx \left\{2 \log a + \log\left(1 - \frac{2}{a} \cos x + \frac{1}{a^2}\right)\right\}$$

$$= 2\pi \log a, \text{ by the former case.}$$

It is difficult to say, *a priori*, in what cases the artifice will succeed; but it is manifest that it is chiefly applicable to circular and logarithmic functions. H. G.

V.—ON THE LINEAR MOTION OF HEAT. PART I.*

THE differential equation which expresses the linear motion of heat in an infinite solid, is

$$\frac{dv}{dt} = \frac{d^2 v}{dx^2},$$

where v is the temperature at the time t, of a point at the distance x from a fixed plane, which, for brevity, may be called the *zero plane*, and the conducting power is taken as unity. Its integral may be put under two forms, one containing an arbitrary function of x, and the other containing two arbitrary functions of t. I propose to deduce the latter of these solutions from the former, and to show, so far as possible, the relation which they bear to one another, with regard to the physical problem.

The first of the solutions referred to is

$$\pi^{\frac{1}{2}} v = \int_{-\infty}^{\infty} d\alpha\, \varepsilon^{-\alpha^2} f(x + 2\alpha t^{\frac{1}{2}}) \ldots\ldots\ldots (1).$$

Let ${}_0v$, v_0, represent the values of v corresponding to $t = 0$, and $x = 0$, respectively. Hence, when $t = 0$, we have

$$\pi^{\frac{1}{2}}\, {}_0v = \int_{-\infty}^{\infty} d\alpha\, \varepsilon^{-\alpha^2} fx,$$

or, since $\displaystyle\int_{-\infty}^{\infty} d\alpha\, \varepsilon^{-\alpha^2} = \pi^{\frac{1}{2}}, \quad {}_0v = fx.$

Hence fx is the function expressing the initial distribution of heat, which therefore is, as it should be, quite arbitrary, and sufficient for determining all the succeeding distributions of the temperature. If, however, the varying temperature of any plane, as for instance the zero plane, be subject to any condition, it is obvious that the initial distribution will cease to be altogether arbitrary, as it alone is sufficient to determine the temperatures at all future times. If, however, the initial distribution be given on the positive side of the zero plane, it is clear that a certain initial distribution on the negative side will enable us to subject the variation of the temperature of the zero plane to any condition we please. By applying this principle, we can determine, in the following manner, the variable temperature of any point

* From a Correspondent.

in the cases; first, when the initial distribution on the positive side being given, the temperature of the zero plane is a given function of the time; and secondly, when the part of the solid on the negative side is removed, and the given initial distribution of temperature on the remaining part is dissipated by radiation across the zero plane, into a medium of constant or varying temperature.

1. Let the conditions be

$${}_0v = \phi x \text{ when } x > 0 \ldots\ldots (2),$$
$$v_0 = \xi t \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Let ψx be the distribution on the negative side, necessary to produce (3), when ϕx is the distribution on the positive side. Hence, when $x > 0$, $fx = \phi x$, and when $x < 0$, $fx = \psi x$, and therefore (1) becomes

$$\pi^{\frac{1}{2}} v = \int_{-\frac{x}{2t^{\frac{1}{2}}}}^{\infty} da\, \varepsilon^{-a^2} \phi(x + 2at^{\frac{1}{2}}) + \int_{-\infty}^{-\frac{x}{2t^{\frac{1}{2}}}} da\, \varepsilon^{-a^2} \psi(x + 2at^{\frac{1}{2}}) \ldots\ldots (4),$$

and (3) gives

$$\pi^{\frac{1}{2}} \xi t = \int_0^{\infty} da\, \varepsilon^{-a^2} \phi(2at^{\frac{1}{2}}) + \int_{-\infty}^{0} da\, \varepsilon^{-a^2} \psi(2at^{\frac{1}{2}})$$
$$= \int_0^{\infty} da\, \varepsilon^{-a^2} \{\phi(2at^{\frac{1}{2}}) + \psi(-2at^{\frac{1}{2}})\}.$$

Hence we must find a function F, such that

$$\pi^{\frac{1}{2}} \xi t = \int_0^{\infty} da\, \varepsilon^{-a^2} F(2at^{\frac{1}{2}}) \ldots\ldots\ldots (5);$$

and, when this is done, we have, for determining ψ,

$$\psi(-x) = Fx - \phi x \ldots\ldots\ldots\ldots (6).$$

To determine F, let, in the first place, ξt be a periodical function of t, and let

$$\xi t = \Sigma\left(A_i \cos \frac{2i\pi t}{p} + B_i \sin \frac{2i\pi t}{p}\right) \ldots\ldots (7).$$

Then, by taking $p = \infty$, any unperiodical function may, by Fourier's theorem, be represented in this form. Hence the problem is reduced to that of representing terms of the form $\cos \dfrac{2i\pi t}{p}$, or $\sin \dfrac{2i\pi t}{p}$, by the definite integral $\int_0^{\infty} da\, \varepsilon^{-a^2} F(2at^{\frac{1}{2}})$. To effect this, let $p = a + \sqrt{\{\pm 2mt \sqrt{(-1)}\}}$, in the first member of the equation

$$\int_{-\infty}^{\infty} dp\, \varepsilon^{-p^2} = \pi^{\frac{1}{2}}.$$

Then, dividing by $\varepsilon^{\mp 2mt\sqrt{(-1)}}$, we have

$$\int_{-\infty}^{\infty} da\, \varepsilon^{-a^2}\, \varepsilon^{-2a\sqrt{(mt)}\,[1\pm\sqrt{(-1)}]} = \pi^{\frac{1}{2}}\, \varepsilon^{\pm 2mt\sqrt{(-1)}}.$$

Hence, by addition and subtraction,

$$\mp\int_{-\infty}^{\infty} da\, \varepsilon^{-a^2}\, \varepsilon^{-2a\sqrt{(mt)}}\, \begin{matrix}\sin\\ \cos\end{matrix}\, \{2a\sqrt{(mt)}\} = \pi^{\frac{1}{2}}\, \begin{matrix}\sin\\ \cos\end{matrix}\, (2mt) \ldots\ldots (a),$$

the upper sign being taken along with the sines, and the lower with the cosines. Changing the sign of $\sqrt{(mt)}$, we have

$$\int_{-\infty}^{\infty} da\, \varepsilon^{-a^2}\, \varepsilon^{2a\sqrt{(mt)}}\, \begin{matrix}\sin\\ \cos\end{matrix}\, \{2a\sqrt{(mt)}\} = \pi^{\frac{1}{2}}\, \begin{matrix}\sin\\ \cos\end{matrix}\, (2mt) \ldots\ldots (b).$$

Hence, by addition,

$$\int_{-\infty}^{\infty} da\, \varepsilon^{-a^2}\, \{\varepsilon^{2a\sqrt{(mt)}} \mp \varepsilon^{-2a\sqrt{(mt)}}\}\, \begin{matrix}\sin\\ \cos\end{matrix}\, \{2a\sqrt{(mt)}\} = 2\pi^{\frac{1}{2}}\, \begin{matrix}\sin\\ \cos\end{matrix}\, (2mt);$$

or, since the multiplier of da remains the same when a is changed into $-a$,

$$\int_{0}^{\infty} da\, \varepsilon^{-a^2}\, \{\varepsilon^{2a\sqrt{(mt)}} \mp \varepsilon^{-2a\sqrt{(mt)}}\}\, \begin{matrix}\sin\\ \cos\end{matrix}\, \{2a\sqrt{(mt)}\} = \pi^{\frac{1}{2}}\, \begin{matrix}\sin\\ \cos\end{matrix}\, (2mt) \ldots (c).$$

Hence, if we put $m = \dfrac{i\pi}{p}$, we have

$$\begin{aligned}\int_0^{\infty} da\, \varepsilon^{-a^2}\, \Sigma\Bigg\{A_i\Big(\varepsilon^{2a\sqrt{\frac{i\pi t}{p}}} + \varepsilon^{-2a\sqrt{\frac{i\pi t}{p}}}\Big) \cos\left(2a\sqrt{\frac{i\pi t}{p}}\right) \\ + B_i\Big(\varepsilon^{2a\sqrt{\frac{i\pi t}{p}}} - \varepsilon^{-2a\sqrt{\frac{i\pi t}{p}}}\Big) \sin\left(2a\sqrt{\frac{i\pi t}{p}}\right)\Bigg\} \\ = \pi^{\frac{1}{2}}\, \Sigma\left(A_i \cos\frac{2i\pi t}{p} + B_i \sin\frac{2i\pi t}{p}\right);\end{aligned}$$

and therefore we have, for the form of the function F,

$$\begin{aligned}Fx = \Sigma\Bigg\{A_i\Big(\varepsilon^{x\sqrt{\frac{i\pi}{p}}} + \varepsilon^{-x\sqrt{\frac{i\pi}{p}}}\Big) \cos\left(x\sqrt{\frac{i\pi}{p}}\right) \\ + B_i\Big(\varepsilon^{x\sqrt{\frac{i\pi}{p}}} - \varepsilon^{-x\sqrt{\frac{i\pi}{p}}}\Big) \sin\left(x\sqrt{\frac{i\pi}{p}}\right)\Bigg\} \ldots\ldots (8).\end{aligned}$$

Now, to satisfy (7),

$$A_i = \frac{2}{p}\int_0^p dt'\xi t' \cos\frac{2i\pi t'}{p}, \quad \text{when } i > 0,$$

$$A_0 = \frac{1}{p}\int_0^p dt'\xi t',$$

$$B_i = \frac{2}{p}\int_0^p dt'\xi t' \sin\frac{2i\pi t'}{p}.$$

Hence, the expression for Fx becomes

$$pFx = \int_0^p dt'\xi t' (\epsilon^0 + \epsilon^{-0})$$

$$+ 2\sum_1^\infty \int_0^p dt'\xi t' \left[\epsilon^{x\sqrt{\frac{i\pi}{p}}} \cos\left\{\sqrt{\frac{i\pi}{p}}\left(x - 2t'\sqrt{\frac{i\pi}{p}}\right)\right\}\right.$$

$$\left. + \epsilon^{-x\sqrt{\frac{i\pi}{p}}} \cos\left\{\sqrt{\frac{i\pi}{p}}\left(x + 2t'\sqrt{\frac{i\pi}{p}}\right)\right\}\right];$$

or, since $(\epsilon^z + \epsilon^{-z}) \cos z = \{\epsilon^{z\sqrt{(-1)}} + \epsilon^{-z\sqrt{(-1)}}\} \cos \{z\sqrt{(-1)}\}$,
and $\qquad (\epsilon^z - \epsilon^{-z}) \sin z = -\{\epsilon^{z\sqrt{(-1)}} - \epsilon^{-z\sqrt{(-1)}}\} \sin \{z\sqrt{(-1)}\}$,

and therefore each term of the series remains the same when i is changed into $-i$,

$$pFx = \sum_{-\infty}^\infty \int_0^p dt'\xi t' \left[\epsilon^{x\sqrt{\frac{i\pi}{p}}} \cos\left\{\sqrt{\frac{i\pi}{p}}\left(x - 2t'\sqrt{\frac{i\pi}{p}}\right)\right\}\right.$$

$$\left. + \epsilon^{-x\sqrt{\frac{i\pi}{p}}} \cos\left\{\sqrt{\frac{i\pi}{p}}\left(x + 2t'\sqrt{\frac{i\pi}{p}}\right)\right\}\right] \ldots. (9).$$

If ξt be not periodical, let $p = \infty$. Then, changing the limits of t' to $-\frac{1}{2}p$ and $\frac{1}{2}p$, instead of 0 and p, and putting $\frac{i\pi}{p} = m$, $\frac{\pi}{p} = dm$, we have

$$\pi Fx = \int_{-\infty}^\infty dm \int_{-\infty}^\infty dt'\xi t' \left[\epsilon^{m^{\frac{1}{2}}x} \cos\{m^{\frac{1}{2}}(x - 2t'm^{\frac{1}{2}})\}\right.$$

$$\left. + \epsilon^{-m^{\frac{1}{2}}x} \cos\{m^{\frac{1}{2}}(x + 2t'm^{\frac{1}{2}})\}\right] \ldots. (10).$$

Hence, if we determine F from (9) or (10), and ψ from (6), the solution of the problem is found by using this result in (4).

Equation (6) shows that $\psi(x)$, the initial distribution on the negative side of the zero plane, is composed of two parts, $-\phi(x)$ and $F(-x)$. The first of these, together with ϕx, on the positive side, would obviously have the effect of retaining the temperature of the zero plane at zero. But, in addition to them, there is the distribution $F(-x)$, on the negative side, which is so determined from (9) or (10), that it alone would have the effect of making the subsequent temperature of the zero plane be ξt. Hence, since the result of the two initial distributions coexisting is equal to the sum of the results in the cases in which they exist separately, it follows that, on the whole, the varying temperature of the zero plane is ξt. Hence we see how it is that, without altering the

initial distribution on the positive side, the initial temperature on the negative side may be so distributed as to make the temperature of the zero plane be ξt.

From (9) and (5), and from (10) and (5), we have the following theorems:

$$\left.\begin{aligned} p\pi^{\frac{1}{2}}\xi t &= \sum_{-\infty}^{\infty}\int_0^p dt'\xi t'\int_0^\infty d\alpha\,\varepsilon^{-\alpha^2}\left[\varepsilon^{2\alpha\sqrt{\frac{i\pi t}{p}}}\cos\left\{2\sqrt{\frac{i\pi}{p}}\left(\alpha t^{\frac{1}{2}} - t'\sqrt{\frac{i\pi}{p}}\right)\right\}\right. \\ &\qquad\qquad \left. + \varepsilon^{-2\alpha\sqrt{\frac{i\pi t}{p}}}\cos\left\{2\sqrt{\frac{i\pi}{p}}\left(\alpha t^{\frac{1}{2}} + t'\sqrt{\frac{i\pi}{p}}\right)\right\}\right] \\ \pi^{\frac{3}{2}}\xi t &= \int_{-\infty}^{\infty} dm\int_{-\infty}^{\infty} dt'\,\xi t'\int_0^\infty d\alpha\,\varepsilon^{-\alpha^2}\left[\varepsilon^{2\alpha\sqrt{(mt)}}\cos\left\{2m^{\frac{1}{2}}\left(\alpha t^{\frac{1}{2}} - t'm^{\frac{1}{2}}\right)\right\}\right. \\ &\qquad\qquad \left. + \varepsilon^{-2\alpha\sqrt{(mt)}}\cos\left\{2m^{\frac{1}{2}}\left(\alpha t^{\frac{1}{2}} + t'm^{\frac{1}{2}}\right)\right\}\right] \end{aligned}\right\} \quad (11),$$

the first or second being used according as ξt is or is not periodical.

These theorems obviously hold when t is negative as well as when it is positive. Hence we have found the distribution on the negative side of the zero plane, which not only produces in every succeeding time the given temperature of the zero plane, but would also follow if, for negative values of t, the temperature had been the same function of these negative values. In general, however, the temperature of any plane except the zero plane, as given by (4), will be impossible for negative values of t, since, except on a particular assumption with respect to ϕx, or the value of ${}_0v$, when x is positive, the initial distribution, represented by ψx and ϕx, is not of such a form as to be any stage, except the first, in a system of varying possible temperatures, or is not producible by any previous possible distribution. Thus, if ${}_0v = 0$ when x is positive, and ${}_0v = F(-x)$ when x is negative, the state represented cannot be the result of any *possible* distribution of temperature which has previously existed, though if in (4) we put $\phi x = 0$, and give t a negative value, we find a distribution, probably impossible except when $x = 0$, which will produce the distribution ${}_0v$, when $t = 0$.

VI.—ON THE INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS.

By B. Bronwin.

In my former paper on this subject, several differential equations were integrated in finite terms, which by the ordinary methods would not have been done without an infinite series. As there is some difficulty in ascertaining the required form of the integral, I shall here exhibit a method by which that may be always done.

Let $$A_0 \frac{d^2y}{dx^2} + B_0 \frac{dy}{dx} + C_0 y = 0.$$

Differentiating and eliminating y, we obtain

$$A_1 \frac{d^3y}{dx^3} + B_1 \frac{d^2y}{dx^2} + C_1 \frac{dy}{dx} = 0.$$

$$A_1 = A_0 C_0,\ \ B_1 = B_0 C_0 + \overset{1}{A}_0 C_0 - A_0 \overset{1}{C}_0,\ \ C_1 = C_0^2 + \overset{1}{B}_0 C_0 - B_0 \overset{1}{C}_0;$$

$$\overset{1}{A}_0 = \frac{dA_0}{dx},\ \ \overset{1}{B}_0 = \frac{dB_0}{dx},\ \ \&c.$$

In like manner, repeating the same operations, we shall have

$$A_2 \frac{d^4y}{dx^4} + B_2 \frac{d^3y}{dx^3} + C_2 \frac{d^2y}{dx^2} = 0,\quad A_3 \frac{d^5y}{dx^5} + B_3 \frac{d^4y}{dx^4} + C_3 \frac{d^3y}{dx^3} = 0;$$

. . . .

$$A_r \frac{d^{r+2}y}{dx^{r+2}} + B_r \frac{d^{r+1}y}{dx^{r+1}} + C_r \frac{d^ry}{dx^r} = 0.$$

Let the following be $A_{r+1} \frac{d^{r+3}y}{dx^{r+3}} + B_{r+1} \frac{d^{r+2}y}{dx^{r+2}} = 0$; the last term having vanished.

If we eliminate $\frac{dy}{dx}$ between the first and second of these, and $\frac{d^2y}{dx^2}$ between that result and the third, and so on; we shall ultimately have $$y = M \frac{d^{r+1}y}{dx^{r+1}} + N \frac{d^{r+2}y}{dx^{r+2}},$$

M and N being finite functions of x. Integrating the final equation of the series, we have

$$\frac{d^{r+2}y}{dx^{r+2}} = aX,\quad \frac{d^{r+1}y}{dx^{r+1}} = a\int X dx + b,$$

X a known function of x, and a and b arbitraries. Substituting these values in the expression of y, we have

$$y = a\,(M\!\int\! X dx + NX) + bM,$$

the complete integral.

If C_0 be constant, M will be an integer function of x; and since $y = bM$ is a particular integral, the proposed equation will give it immediately by series. The integral $\int X dx$ must be reduced to the simplest form possible. Then, putting $M\!\int\! X dx + NX$ for y in the proposed, we shall have N by series. This is the easiest way of obtaining M and N when C_0 is constant; but if it be a function of x, these quantities are fractions, and can only be obtained by elimination as before explained. By this method we should easily find the second particular integrals

$$y = uc^{qx} + v\int \frac{dx}{x}\, c^{qx}, \quad y = uc^{\frac{1}{2}q^2x^2} + v\int dx c^{\frac{1}{2}q^2x^2}$$

given in my former paper.

Let us now illustrate the method by one or two examples.

Suppose $$(a + bx)\frac{d^2y}{dx^2} + (f + gx)\frac{dy}{dx} - gy = 0;$$

differentiate, and $$(a + bx)\frac{d^3y}{dx^3} + (b + f + gx)\frac{d^2y}{dx^2} = 0.$$

Therefore making $\beta = \dfrac{ag}{b^2} - \dfrac{f}{b} - 1, \quad \gamma = \dfrac{g}{b}$, we have

$$\frac{d^2y}{dx^2} = C\,(a + bx)^{\beta}\, c^{-\gamma x}, \quad \frac{dy}{dx} = C\!\int\! dx c^{-\gamma x}\,(a + bx)^{\beta} + C \text{ and}$$

$$y = C\{c^{-\gamma x}(a + bx)^{\beta+1} + (f + gx)\int dx c^{-\gamma x}\,(a + bx)^{\beta}\} + C'\,(f + gx);$$ the complete integral. Constants multiplying the arbitraries are omitted, as it is only changing those arbitraries.

Let $$(a + bx)\frac{d^2y}{dx^2} + (f + gx)\frac{dy}{dx} - 2gy = 0;$$

we have, by differentiation,

$$(a + bx)\frac{d^3y}{dx^3} + (b + f + gx)\frac{d^2y}{dx^2} - g\frac{dy}{dx} = 0;$$

$$(a + bx)\frac{d^4y}{dx^4} + (2b + f + gx)\frac{d^3y}{dx^3} = 0.$$

By elimination

$$2g^2y = (a + bx)(f + gx)\frac{d^3y}{dx^3} + \{g\,(a + bx) + (f + gx)(b + f + gx)\}\frac{d^2y}{dx^2}.$$

By integration

$$\frac{d^3y}{dx^3} = Cc^{-\gamma x}(a+bx)^\beta, \quad \frac{d^2y}{dx^2} = C\int dx c^{-\gamma x}(a+bx)^\beta + C',$$

where $\beta = \frac{ag}{b^2} - \frac{f}{b} - 2$, $\gamma = \frac{g}{b}$. Substituting the values of $\frac{d^3y}{dx^3}$, $\frac{d^2y}{dx^2}$ in that of $2g^2y$, we have the complete integral.

To give an example or two in which C_0 is not constant; let

$$(a + bx + ex^2 + fx^3)\frac{d^2y}{dx^2} + (g + hx + kx^2)\frac{dy}{dx} + (l - kx)y = 0;$$

where $k = -(h+l)\frac{l}{g}$. After one differentiation and elimination of y, $\frac{dy}{dx}$ also vanishes, and we obtain an equation of the form

$$P\frac{d^3y}{dx^3} + Q\frac{d^2y}{dx^2} = 0;$$

which gives

$$\frac{d^2y}{dx^2} = C\frac{l - kx}{a + bx + ex^2 + fx^3} \cdot c^{-\int\frac{(g+hx+kx^2)dx}{a+bx+ex^2+fx^3}} = CX;$$

$$\frac{dy}{dx} = C\int Xdx + C'; \text{ and}$$

$$y = C\left\{\frac{a + bx + ex^2 + fx^3}{l - kx}X + \left(\frac{g}{l} - x\right)\int Xdx\right\} + C'\left(\frac{g}{l} - x\right).$$

Let $$A_0\frac{d^2y}{dx^2} + (a - x)C_0\frac{dy}{dx} + C_0y = 0;$$

differentiating, we have

$$A_0\frac{d^3y}{dx^3} + \left\{\left(\frac{A_0}{C_0}\right)^1 + a - x\right\}C_0\frac{d^2y}{dx^2} = 0; \quad \frac{d^2y}{dx^2} = k\frac{C_0}{A_0}c^{-\int\frac{C_0}{A_0}(a-x)dx};$$

$$\frac{dy}{dx} = k\int\frac{C_0}{A_0}dxc^{-\int\frac{C_0}{A_0}(a-x)dx} + l;$$

k and l being arbitraries. Therefore

$$y = k\left\{c^{-\int\frac{C_0}{A_0}(a-x)dx} + (a-x)\int\frac{C_0}{A_0}dxc^{-\int\frac{C_0}{A_0}(a-x)dx}\right\} + l(a-x),$$

where k and l are changed into $-k$ and $-l$.

Without the last term vanishing, if we arrive at an equation which we can integrate, we shall obtain the integral of the proposed in the same manner. And it is obvious that the

method will apply to equations of the third order. If by successive differentiation we arrive at an equation, the two last terms of which vanish; or if only the last term vanish, and we can integrate the resulting equation of the second order; we shall obtain the integral of the proposed.

The equation $\dfrac{d^{r+2}y}{dx^{r+2}} = aX$ would give

$$y = a\int^{r+2} X dx^{r+2} + c_0 + c_1 x + c_2 x^2 + \ldots\ldots + c_{r+1} x^{r+1},$$

which is the more complex of the two particular integrals. It may be observed that $a\int^{r+2} X dx^{r+2}$ is the remainder of the series after the $r+2$ first terms. Suppose this particular integral developed by Taylor's or Maclaurin's Theorem, we shall have

$$y = \Sigma a_n x^n + \frac{1}{P(n)} \int_0^h \frac{d^{n+1}y}{dx^{n+1}} (h-x)^n dx,$$

h afterwards being changed into x.

Let $(1-x^2)\dfrac{d^2y}{dx^2} + mx\dfrac{dy}{dx} - ry = 0$; $m = p + q - 1$, $r = pq$.

Then $$(1-x^2)\frac{d^3y}{dx^3} + m_1 x \frac{d^2y}{dx^2} - r_1 y = 0;$$

$$m_1 = p_1 + q_1 - 1, \quad r_1 = p_1 q_1, \quad p_1 = p - 1, \quad q_1 = q - 1.$$

Hence, continuing the process, if p be integer, and we make $q = p + t$; we find

$$(1-x^2)\frac{d^{p+2}y}{dx^{p+2}} + (t-1)x\frac{d^{p+1}y}{dx^{p+1}} = 0, \quad \frac{d^{p+1}y}{dx^{p+1}} = C(1-x^2)^{\frac{t-1}{2}}.$$

Therefore the remainder is

$$\frac{C}{P(p)}\int (1-x^2)^{\frac{t-1}{2}} (h-x)^p dx.$$

Or, changing x into vx, and h into x, it is

$$C\frac{x^{p+1}}{P(p)}\int_0^1 (1-x^2v^2)^{\frac{t-1}{2}}(1-v)^p dv, \text{ or } C\frac{x^{p+1}}{P(p)}\int_0^1 \{1-x^2(1-u)^2\}^{\frac{t-1}{2}} u^p du.$$

This equation is integrated in my former paper; one of the series terminates in the case supposed, and the above is obviously the remainder of the other which does not terminate. The first term of the above remainder expanded is $\dfrac{Cx^{p+1}}{P(p)}$. This compared with the first term of the continuation of the series will give C. Thus, after finding the series in the

ordinary way, we shall find the remainder after the term $a_p x^p$, by a definite integral, which is easier than to integrate $\frac{d^{p+1}y}{dx^{p+1}} = aX$, $p+1$ times, and then to substitute the result in the proposed, in order to determine the $p+1$ arbitraries. The two methods, however, amount in reality to the same thing.

Again, let $\frac{d^2z}{dx^2} - q^2x\frac{dz}{dx} + q^2mz = 0$, m an integer. We find, as before,

$$\frac{d^{m+2}z}{dx^{m+2}} - q^2x\frac{d^{m+1}z}{dx^{m+1}} = 0, \quad \text{and} \quad \frac{d^{m+1}z}{dx^{m+1}} = Cc^{\frac{1}{2}q^2x^2}.$$

In this case, therefore, the remainder is

$$\frac{C}{P(m)}\int dx\, c^{\frac{1}{2}q^2x^2}(h-x)^m\, dx, \quad \text{or} \quad \frac{Cx^{m+1}}{P(m)}\int dv\, c^{\frac{1}{2}q^2x^2v^2}(1-v)^m.$$

This is always reducible to $w\,c^{\frac{1}{2}q^2x^2} + u\int dv\, c^{\frac{1}{2}q^2x^2v^2}$, w and u being finite integral functions of x. This is the remainder of the series which expresses that particular integral of the proposed which does not terminate, the series being carried to the term $a_m x^m$.

We may employ successive integration, integrating every term by parts. This will apply where the former method does not. Supposing that our equation may have the last term destroyed by $r+1$ integrations, let

$$A_0\frac{d^2y}{dx^2} + B_0\frac{dy}{dx} + C_0y = 0.$$

Then $A_0\frac{dy}{dx} + P_0y + \int Q_0y\,dx + a = 0$, a an arbitrary. Make $y_1 = \int Q_0y\,dx + a$; and by substitution the last becomes

$$A_1\frac{d^2y_1}{dx^2} + B_1\frac{dy_1}{dx} + C_1y_1 = 0.$$

Continuing this process, we have at length

$$A_r\frac{d^2y_r}{dx^2} + B_r\frac{dy_r}{dx} + C_ry_r = 0, \qquad A_r\frac{dy_r}{dx} + P_ry_r = C,$$

the integral $\int Q_ry\,dx$ vanishing. This integrated gives

$$y_r = Cc^{-\int\frac{P_r}{A_r}dx}\int\frac{dx}{A_r}\,c^{\int\frac{P_r}{A_r}dx} + C'c^{-\int\frac{P_r}{A_r}dx}.$$

Then we have

$$y = \frac{1}{Q_0}\,\frac{dy_1}{dx}, \quad y_1 = \frac{1}{Q_1}\,\frac{dy_2}{dx}, \quad \&c.;$$

and $$y = \frac{1}{Q_0} \cdot \frac{d}{dx}\left\{\frac{1}{Q_1} \cdot \frac{d}{dx}\left(\frac{1}{Q_2} \cdots \frac{dy_r}{dx}\right)\right\}.$$

As an example,

let $$\frac{d^2y}{dx^2} + (a + 2bx)\frac{dy}{dx} + 4by = 0.$$

After two integrations, we have

$$\frac{dy_1}{dx} + (a + 2bx)\, y_1 = C, \quad y_1 = \int y dx + a, \quad y = \frac{dy_1}{dx}.$$

By integration

$$y_1 = Cc^{-ax-bx^2} \int dx c^{ax+bx^2} + C'c^{-ax-bx^2};$$

and hence, if $M = ax + bx^2$,

$$y = C\left\{(a + 2bx)\, c^{-M} \int dx c^{M} - 1\right\} + C'(a + 2bx)\, c^{-M}.$$

If $\dfrac{d^2y}{dx^2} + (a + 2bx)\dfrac{dy}{dx} + 2rby = 0$; we have, obviously,

$\dfrac{dy_r}{dx} + (a + 2bx)\, y_r = C$, and $y_r = Cc^{-ax-bx^2} \int dx c^{ax+bx^2} + C'c^{-ax-bx^2}$

Here we should have r differentiations to perform to find y. This might be impracticable. We should therefore make $y = zc^{-ax-bx^2}$, and we have

$$\frac{d^2z}{dx^2} - (a + 2bx)\frac{dz}{dx} + 2(r - 1)\, by = 0.$$

This will give a particular integral by descending series, viz. $z = a_{r-1}\, x^{r-1} + a_{r-2}\, x^{r-2} + \ldots\ldots + a_0$. Let this be called v, and make $z = uc^{ax+bx^2} + v\int dx c^{ax+bx^2}$. Substituting this value for z in the above equation, we shall determine u in the same manner as in several examples of my former paper.

The modes we have employed very nearly approach to certain transformations, which we will briefly notice.

Let $$x\left(\frac{d^2y}{dx^2} + r^2y\right) + q\frac{dy}{dx} = 0.$$

Make $$\frac{d^2y}{dx^2} + r^2y = y;\ \text{and we have}$$

$$x\left(\frac{d^2y_1}{dx^2} + r^2y_1\right) + (q + 2)\frac{dy_1}{dx} = 0.$$

Make $x\dfrac{dy}{dx} + (q - 1)\, y = y_1$; and the same equation becomes

$$x\left(\frac{d^2y_1}{dx^2} + r^2y_1\right) + (q - 2)\frac{dy_1}{dx} = 0.$$

Thus we can increase or diminish q by any multiple of 2.

Let $$\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + ray = 0.$$

Make $\dfrac{dy}{dx} + axy = y_1$; and the transformed is

$$\frac{d^2y_1}{dx^2} + ax\frac{dy_1}{dx} + (r-1)\,ay_1 = 0,$$

where the coefficient of the last term is diminished of unity, and by repeating the process, if r be integer, may be destroyed. But we have already integrated this.

Let $$x^2\left(\frac{d^2y}{dx^2} + q^2y\right) = my; \quad m = r(r+1).$$

Make $\dfrac{dy}{dx} + \dfrac{r}{x}y = y_1$. Then we find

$$x^2\left(\frac{d^2y_1}{dx^2} + q^2y_1\right) = m_1y_1; \quad m_1 = (r-1)\,r,$$

and, consequently, r is diminished of a unit, and by a repetition of the process the last term may be destroyed.

Let $$x\left(\frac{d^2y}{dx^2} - q^2y\right) = 2rqy.$$

Make $y = zc^{\pm qx}$; and there results

$$x\frac{d^2z}{dx^2} \pm 2qx\frac{dz}{dx} = 2rqz.$$

A particular integral of each of these, $\pm r$ being integer, has been found in finite terms in the former paper. They will give the complete integral of the proposed.

If $x\dfrac{d^2y}{dx^2} + \frac{1}{2}\dfrac{dy}{dx} + by = 0$, $y = Cc^{2b^{\frac{1}{2}}x^{\frac{1}{2}}\sqrt{(-1)}} + C'c^{-2b^{\frac{1}{2}}x^{\frac{1}{2}}\sqrt{(-1)}}$.

Therefore $$x\frac{d^2y}{dx^2} + (\tfrac{1}{2} \pm i)\frac{dy}{dx} + by = 0,$$

i an integer, is always integrable in finite terms.

Many of the equations treated of in these papers seem particularly adapted for the application of a definite integral. There is, however, great difficulty in that application; and in some cases I have been unable to apply it. I will put down a few examples where it succeeds, but the form has been, for the most part, difficult to find.

Let $\dfrac{d^2y}{dx^2} + q^2x\dfrac{dy}{dx} + mq^2y = 0$. The integral is

$$y = C\int_0^\infty c^{-\frac{1}{2}t^2}t^{m-1}\,dt\,\sin(qxt) + C'\int_0^\infty c^{-\frac{1}{2}t^2}t^{m-1}\,dt\,\cos(qxt).$$

When m is integer, one of these may be integrated in finite terms.

Let $$x\frac{d^2y}{dx^2} + 2q\frac{dy}{dx} + r^2xy = 0,$$

q between the limits 0 and 1,

$$y = C\int_0^1 (1-t^2)^{q-1}\,dt\cos(rxt) + C'x^{1-2q}\int_0^1 (1-t^2)^{-q}\,dt\cos(rxt).$$

These, by expanding $\cos(rxt)$, and integrating, will give the forms obtained by series.

Mr. Hymers (see *Differential Equations*, p. 84) is mistaken in supposing that $u = \beta\int(t^2-n^2)^m\,dt\cos(xt+\alpha)$ is a complete integral.

It reduces to $A\int(t^2-n^2)^m\,dt\cos(xt) + B\int(t^2-n^2)^m\,dt\sin(xt)$. The latter of these integrals is zero, the negative and affirmative parts destroying one another.

Supposing q positive, the complete integral of

$$x\frac{d^2y}{dx^2} + 2q\frac{dy}{dx} + r^2xy = 0,$$

is $y = C\int_0^1 t^{q-1}dt(1-t)^{q-1}\cos(2rxt-rx) + C'\int_0^1 t^{q-1}dt(1-t)^{q-1}\sin(2rxt-rx)$;

and that of $$x\frac{d^2y}{dx^2} + 2q\frac{dy}{dx} - r^2xy = 0,$$

is $y = Cc^{rx}\int_0^1 t^{q-1}\,dt(1-t)^{q-1}c^{-2rxt} + C'c^{-rx}\int_0^1 t^{q-1}\,dt(1-t)^{q-1}c^{2rxt}$.

From what has been done in these papers, it appears that we may sometimes, by series, immediately obtain a particular integral in finite terms, and then by transforming the equation we may obtain the other in finite terms also; and sometimes when we can obtain neither, by transformation we shall have both in finite terms. There is one circumstance respecting certain equations which might perplex, and which therefore requires to be noticed. The equation $x^2\frac{d^2y}{dx^2} - m\frac{dy}{dx} - ry = 0$, $r = p(p-1)$ (see former paper) gives immediately $y = Cv$, a series that terminates; and then, making $y = zc^{-\frac{m}{x}}$, we find $z = C'w$ in finite terms, both by descending series. But if we develop $wc^{-\frac{m}{x}}$ we find it to give v. The fact is that the series breaks, certain of the terms vanish; it then goes on again; and the continuation is really the second integral of the proposed, which is only a continuation of the series which gave the first, after certain intermediate terms have disappeared.

VII.—ON ELIMINATION.

By R. Moon, M.A. Fellow of Queens' College.

We purpose, in the following paper, to indicate an easy practical method of obtaining the principal or symmetrical factor of the result of elimination between two functions of the same number of dimensions.

The result of the elimination of x from the quantities

$$ax + b,$$
$$\alpha x + \beta,$$
$$\text{is} \quad a\beta - \alpha b.$$

To find the principal factor R of the result of elimination between

$$ax^2 + bx + c,$$
$$\alpha x^2 + \beta x + \gamma.$$

Suppose c and α each $= 0$, the result of elimination between

$$ax + b,$$
$$\beta x + \gamma,$$
$$\text{is} \quad a\gamma - b\beta.$$

Multiply this by $(a\gamma)$, and we have

$$a^2\gamma^2 - ab\,\beta\gamma.$$

$$\text{For} \quad a^2\gamma^2 \text{ put } (a\gamma - \alpha c)^2,$$
$$ab\,\beta\gamma \;\ldots\; (a\beta - \alpha b)(b\gamma - \beta c),$$

and we have

$$R = (a\gamma - \alpha c)^2 - (a\beta - \alpha b)(b\gamma - \beta c).$$

Let R be the principal factor of the result of elimination between

$$ax^3 + bx^2 + cx + d,$$
$$\alpha x^3 + \beta x^2 + \gamma x + \delta;$$

the result of elimination between

$$ax^2 + bx + c,$$
$$\beta x^2 + \gamma x + \delta,$$

$$\text{is} \quad (a\delta - \beta c)^2 - (a\gamma - \beta b)(b\delta - \gamma c),$$
$$\text{or} \quad a^2\delta^2 + \beta^2c^2 - 2a\delta\,\beta c$$
$$- a\gamma\, b\delta + a\gamma^2 c + b^2\beta\delta - b\beta\,\gamma c;$$

or, multiplying by $(a\delta)$,

$$a^3\delta^3 + a\delta\beta^2c^2 - 2a^2\delta^2\beta c - a^2\gamma b\delta^2$$
$$+ a^2\gamma^2 c\delta + ab^2\beta\delta^2 - ab\beta c\gamma\delta.$$

$$
\begin{array}{llll}
\text{For} & +a^3\delta^3 & \text{put} & +(a\delta-\alpha d)^3 \\
& +ab^2\beta\delta^2 & \ldots & +(a\beta-\alpha b)(b\delta-\beta d)^2 \\
& +a^2c\gamma^2\delta & \ldots & +(a\gamma-\alpha c)^2(c\delta-\gamma d) \\
& -a^2b\gamma\delta^2 & \ldots & -(a\gamma-\alpha c)(b\delta-\beta d)(a\delta-\alpha d) \\
& -2a^2\beta c\delta^2 & \ldots & -2(a\beta-\alpha b)(c\delta-\gamma d)(a\delta-\alpha d)
\end{array}
$$

$+a\beta^2c^2\delta-ab\beta\gamma c\delta$, or

$$+(a\beta c\delta)(\beta c-b\gamma)\ \ldots\ -(a\beta-\alpha b)(c\delta-\gamma d)(b\gamma-\beta c);$$

$$
\begin{aligned}
\therefore\ R=(a\delta-\alpha d)^3 &+(a\gamma-\alpha c)^2(c\delta-\gamma d) \\
&+(a\beta-\alpha b)(b\delta-\beta d)^2 \\
&-(a\gamma-\alpha c)(b\delta-\beta d)(a\delta-\alpha d) \\
&-2(a\beta-\alpha b)(a\delta-\alpha d)(c\delta-\gamma d) \\
&-(a\beta-\alpha b)(c\delta-\gamma d)(b\gamma-\beta c).
\end{aligned}
$$

It may be observed, that had we treated $a\beta^2c^2\delta$ in the same way as the quantities which preceded it, we should have had either $\quad -(a\beta-\alpha b)(b\gamma-\beta c)(c\delta-\gamma d)$
or else $\quad (a\delta-\alpha d)(b\gamma-\beta c)^2$.

Neither of which quantities reduce themselves to $a\beta^2c^2\delta$, when $\alpha=0$, $d=0$; and the same may be said of $ab\beta\gamma c\delta$, whereas it will be found that each of the quantities as above substituted, *will* reduce itself to its corresponding quantity when $\alpha=0$, $d=0$.

By the help of this last remark, we may proceed to the case of four dimensions, of which we shall however only exhibit the result, which is, if we put $a\beta-\alpha b=f(a\,B)$ similarly for the other quantities,

$$
\begin{aligned}
&+\{f(a\epsilon)\}^4+\{f(a\gamma)\}^2\{f(c\epsilon)\}^2 \\
&-\{f(a\epsilon)\}^2\{f(a\delta)f(b\epsilon)+2f(a\gamma)f(c\epsilon)+3f(a\beta)f(d\epsilon)\} \\
&-\{f(a\gamma)\}^2\{f(c\delta)f(d\epsilon)+2f(b\epsilon)f(d\epsilon)\} \\
&-f(a\gamma)f(c\epsilon)\{f(a\delta)f(b\epsilon)+f(a\beta)f(d\epsilon)\} \\
&-\{f(c\epsilon)^2\}\{f(a\beta)f(b\gamma)+2f(a\beta)f(a\delta)\} \\
&+f(a\gamma)f(a\delta)f(b\epsilon)f(d\epsilon) \\
&+f(c\epsilon)f(a\beta)f(b\delta)f(b\epsilon) \\
&-\{f(a\delta)^3\}f(d\epsilon) \\
&-\{f(b\epsilon)^3\}f(a\beta) \\
&+f(a\delta)f(b\epsilon)f(a\beta)f(d\epsilon) \\
&+f(c\delta)f(b\gamma)f(a\beta)f(d\epsilon) \\
&+2f(a\delta)f(c\delta)f(a\beta)f(d\epsilon) \\
&+2f(b\epsilon)f(b\gamma)f(a\beta)f(d\epsilon) \\
&+f(a\epsilon)f(c\epsilon)\{f(a\delta)^2+3f(a\beta)f(b\epsilon)\} \\
&+f(a\epsilon)f(a\gamma)\{f(b\epsilon)^2+3f(a\delta)f(d\epsilon)\}.
\end{aligned}
$$

VIII.—EVALUATION OF CERTAIN DEFINITE INTEGRALS.

By R. L. Ellis, B.A. Fellow of Trinity College.

When the value of a definite integral is known, we may, if it involve an arbitrary parameter, integrate it (under certain conditions) with respect to this quantity. The result thus obtained involves an arbitrary constant of integration; in order to eliminate it, we may ascribe two different values to the quantity for which the integration has been effected, and then, of the two corresponding equations thus got, subtract one from the other. In other words, we integrate between limits for the arbitrary parameter, and thus get a new definite integral, involving two arbitrary quantities, namely, the two limiting values ascribed to the single one involved in the original integral. We may integrate again, with respect to either of these, and so on. But this method of proceeding, though it will lead to a variety of particular results, is not well fitted to show the nature of the class of definite integrals to which they all belong, and which may be obtained by repeated integrations for an arbitrary parameter.

If we integrate n times successively, we shall introduce n constants. These may be eliminated *at once*, in the manner I am about to point out. The result thus got, includes for every original definite integral, all that can be deduced from it by n integrations for an arbitrary parameter.

The following theorem will serve to illustrate the general method.

If Fx is a rational and integral function of circular functions of x, (sines and cosines), then we may express in finite terms the value of $\int_{-\infty}^{+\infty} \frac{Fx}{x^n}\, dx$, n being a positive integer, and such that $\left.\frac{Fx}{x^n}\right\}_0$ is not infinite.

This theorem applies to several remarkable definite integrals, some of which occur in the theory of probabilities; there are others which do not seem to have been noticed.

Dem. Fx, as every function of x, may be considered the sum of two functions, one of which remains unchanged when x changes its sign, and the other changes its sign with that of x; its value *aux signes près* remaining unaltered. Hence, whether n is odd or even, we may write

$$\frac{Fx}{x^n} = \frac{fx}{x^n} + \frac{\phi x}{x^n},$$

where $$\frac{f(-x)}{(-x)^n} = \frac{fx}{x^n}, \quad \text{and} \quad \frac{\phi(-x)}{(-x)^n} = -\frac{\phi x}{x^n}.$$

It is obvious that

$$\int_{-\infty}^{+\infty} \frac{Fx}{x^n}\, dx = 2\int_0^{\infty} \frac{fx}{x^n}\, dx;$$

and that if n is odd, fx, which is of course a rational and integral function of circular functions (sines and cosines) of x, must be developable in a series of sines exclusively, and if n is even in a series of cosines exclusively.

Thus, we may assume

$$fx = \Sigma A \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} ax \ldots\ldots\ldots\ldots\ldots (1).$$

Now, as $\dfrac{fx}{x^n}$ is not infinite when $x = 0$, the lowest power of x which can enter into fx must be not $< n$, call it m, and develope in powers of x, every sine or cosine which appears on the second side of the last written equation. We must have $\Sigma A a^{m-2} = 0$, $\Sigma A a^{m-4} = 0$, &c.... $\frac{1}{2}m - 1$ equations if m is even, and $\frac{1}{2}(m-1)$ if it is odd.

Let us now consider the definite integral

$$\int_0^{\infty} e^{-ax} \cos rx dx = \frac{a}{a^2 + r^2}.$$

Integrating it repeatedly for r, we get

$$\int_0^{\infty} e^{-ax} \frac{\sin rx}{x}\, dx = \tan^{-1}\frac{r}{a}$$

$$\int_0^{\infty} e^{-ax} \frac{\cos rx}{x^2}\, dx = -r \tan^{-1}\frac{r}{a} + a \log\sqrt{(a^2 + r^2)} + C,$$

and generally

$$\int_0^{\infty} e^{-ax} \frac{\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} rx}{x^n}\, dx = \pm \frac{r^{n-1}}{[n-1]} \tan^{-1}\frac{r}{a} + aF_1(ra)$$
$$+ Cr^{n-3} + C_1 r^{n-5} + \&c. \ldots\ldots\ldots\ldots (2),$$

where $F_1(ra)$ does not become infinite for $a = 0$.

Replace r by every quantity represented in (1) by the general symbol a. Multiply each result by the corresponding coefficient A, and add.

Then, in virtue of the conditions,

$$\Sigma A a^{m-2} = 0, \quad \Sigma A a^{m-4} = 0, \ \&c.$$

we shall have

$$\int_0^{\infty} e^{-ax} \frac{fx}{x^n}\, dx = \pm \Sigma \frac{\Sigma A a^{n-1}}{[n-1]} \tan^{-1}\frac{a}{a} + a\Sigma A F_1(aa).$$

Put $a = 0$, then $\tan^{-1}\dfrac{a}{a} = \pm\dfrac{\pi}{2}$, according to whether a is $>$ or $<$ than zero.

Thus we have

$$\int_0^\infty \frac{fx}{x^n}\,dx = \pm \frac{\pi}{2\,[n-1]}\,\Sigma \pm A a^{n-1} \ldots\ldots\ldots\ (3).$$

From hence, the truth of our theorem is obvious.

Of the ambiguous signs outside the symbol of summation, the upper is to be taken when n is of the forms $4p$, or $4p + 1$.

When a is positive, we must take the upper of the ambiguous signs under the Σ.

It will be remarked, that in obtaining (3) we have eliminated all the constants at once, instead of getting rid of them one by one by particular conditions at each successive integration, and that the generality of this method enables us to recognize a class of definite integrals, which are all deduced from the known value of $\int_0^\infty e^{-ax} \cos rx\, dx$.

Equation (3) admits of several remarkable applications. Thus let us suppose $n = 3$ and $fx = \sin ax \sin bx \sin cx$: then

$$fx = -\tfrac{1}{4}\{\sin(a+b+c)x - \sin(-a+b+c)x - \sin(a-b+c)x - \sin(a+b-c)x\}.$$

Consequently, we have by (3)

$$\int_0^\infty \frac{\sin ax \sin bx \sin cx}{x^3}\,dx = \frac{\pi}{4}\{s^2 \mp (s-a)^2 \mp (s-b)^2 \mp (s-c)^2\}$$

where, as in trigonometrical formulæ,

$$2s = a + b + c:$$

the upper sign is to be taken when the quantity to which it is affixed is > 0.

Likewise $\displaystyle\int_0^\infty \frac{\sin ax \sin bx \sin cx}{x}\,dx = \frac{\pi}{8}(1 \mp 1 \mp 1 \mp 1)$

where the signs follow the same rule as in the former case; the different unities involved being the zero powers of s, $s - a$, &c.

Let us now suppose that $fx = \sin^m x \cos zx$; the corresponding integral, viz. $\displaystyle\int_0^\infty \frac{\sin^m x \cos zx}{x^n}\,dx$ occurs in the theory of probabilities. Its value is given at p. 170 of *Laplace's Théorie des Probabilities*, where it is obtained by a method founded on a transition from real to imaginary quantities. The nature of what are called imaginary quantities is certainly better understood than it was some time since; but it seems to have been the opinion of Poisson, as well as of Laplace himself, that results thus obtained require confirmation. In this view I confess I do not acquiesce; but if only in deference

to their authority, it may be desirable to show how readily imaginary quantities may be avoided in estimating the value of the integral in question.

$$\sin^m x = \pm \frac{1}{2^{m-1}}\left\{\cos mx - \frac{m}{1}\cos(m-2)x + \&c.\right\} \text{ if } m \text{ is odd,}$$

$$\text{and } = \pm \frac{1}{2^{m-1}}\left\{\sin mx - \frac{m}{1}\sin(m-2)x + \&c.\right\} \text{ if it is even.}$$

Hence, by (3), $\displaystyle\int_0^\infty \frac{\sin^m x \cos zx}{x^n}\,dx =$

$$\frac{\pi}{[n-1]\,2^m}\left\{(m+z)^{n-1} \pm (m-z)^{n-1} - \frac{m}{1}\{(m+z-2)^{n-1} \pm (m-z-2)^{n-1}\} + \&c.\right\}$$

Let us suppose $z > m$; then the lower of each pair of ambiguous signs must be taken, and the expression within the brackets may be written thus

$$(m+z)^{n-1} - \frac{m}{1}(m+z-2)^{n-1} + \&c. + \frac{m}{1}(m-z-2)^{n-1} - (m-z)^{n-1} \ldots\ldots (q).$$

As, from the nature of the case, m and n are either both odd or both even, if m is even, $n-1$ is odd, and therefore $(m-z)^{n-1} = -(z-m)^{n-1}$; and thus, in every case, (q) equals

$$(1 - D^{-1})^m (z+m)^{n-1} \ldots\ldots \{D\phi z = \phi(z+2) \quad \text{say,}\}$$

and this is $\quad \Delta^m D^{-m}(z+m)^{n-1} = \Delta^m (z-m)^{n-1} = 0,$

since m is $> n-1$. Consequently

$$\int_0^\infty \frac{\sin^m x}{x^n}\cos zx\,dx = 0, \text{ when } z \text{ is } > m,$$

a remark not made by Laplace; when $m = n = 1$, its truth is known.

As $(q) = 0$, add it, multiplied by $\dfrac{\pi}{[n-1]\,2^{m-1}}$, to the value of the integral already found,

$$\therefore \int_0^\infty \frac{\sin^m x}{x^n}\cos zx\,dx = \frac{\pi}{[n-1]\,2^m}\left\{(m+z)^{n-1} - \frac{m}{1}(m+z-2)^{n-1} + \&c.\right\} \ldots (4),$$

where the series stops whenever the next term would introduce a negative quantity raised to the power $n-1$. This is easily seen to be true, for every such term will have a different sign in (q), and in the definite integral, and thus on addition, all such terms will disappear. Equation (4) is Laplace's form; the discontinuity of the function is now expressed, not by ambiguous signs but by the stopping short of the series at different points.

By a similar method, we find that

$$\int_0^\infty \frac{\sin^m x}{x^{n+1}} \sin zx dx = \frac{\pi}{[n]\,2^{m-1}}\left\{(m+z)^{n-1} - \frac{m}{1}(m+z-2)^{n-1} + \&c.\right\}..(5),$$

which might have been deduced from (4) by integrating both sides without introducing any complementary quantities. This remark is general; having once established the general form of (4) for any given value of n, we may deduce from it that which corresponds to any other value of n, simply by differentiation or by integration, without bringing in any constants. I conceive that this remark is general, and if so we may differentiate on both sides with fractional indices. Let the index be $-p$, then as

$$d^p \cos zx = x^p \cos\left(zx + p\,\frac{\pi}{2}\right) dz^p,$$

the first side of (4) will become

$$\int_0^\infty \frac{\sin^m x}{x^n}\, x^{-p} \cos\left(zx - p\,\frac{\pi}{2}\right) dx\,;$$

and the second will be

$$\frac{\pi}{[n-1+p]\,2^m}\left\{(m+z)^{n-1+p} - \&c.\right\}$$

Thus, we get

$$\int_0^\infty \frac{\sin^m x}{x^n}\, x^{-p} \cos\left(zx - p\,\frac{\pi}{2}\right) dx =$$

$$\frac{\pi}{[n-1+p]\,2^m}\left\{(m+z)^{n-1+p} - \frac{m}{1}(m+z-2)^{n-1+p} + \&c.\right\}....(6).$$

If we take $n = m$ this is equivalent to Laplace's general formula (p), at p. 168 of the Théorie.

The method of this paper leads to some elegant results when applied to the definite integral $\int_0^\infty e^{-ax^2}dx$, but it is enough to point out this application, which involves no difficulty whatever.

IX.—PROPOSITIONS IN THE THEORY OF ATTRACTION.

Let x, y, z, be the co-ordinates of any point P, in an attracting or repelling body M; let dm be an element of the mass, at the point P, which will be positive or negative according as it is attractive or repulsive; let x', y', z' be the co-ordinates of an attracted point P';

$$\text{let } \Delta = \{(x' - x)^2 + (y' - y)^2 + (z' - z)^2\}^{\frac{1}{2}};$$

and let
$$v' = \int \frac{dm}{\Delta},$$

the integral including the whole of M. This expression has been called by Green, the potential of the body M, on the point P', and the same name has been employed by Gauss, (in a Mémoire on "General Theorems relating to Attractive and Repulsive Forces," in the Resultate aus den Beobachtungen des magnetischen Vereins im Jahre 1839, Leipsic 1840, edited by M. Gauss and Weber.*) By a known theorem, the components of the attraction of M on P', in the directions of x, y, z, are
$$-\frac{dv'}{dx'}, -\frac{dv'}{dy'}, -\frac{dv'}{dz'},$$
and, if $d\gamma'$ be the element of any line, straight or curved, which passes through P', the attraction in the direction of this element is $\frac{dv'}{d\gamma'}$. Hence it follows that, if a surface be drawn through any point P' for every point of which the potential has the same value, the attraction on every point in the surface is wholly in the direction of the normal. Surfaces for which the potential is constant, are therefore called, by Gauss, *surfaces of equilibrium*. It has been shown in a former paper,† that, if M, instead of an attractive mass, were a group of sources of heat or cold, in the interior of an infinite homogeneous solid, v' would be the permanent temperature produced by them, at P'. In that case, the surfaces of equilibrium would be *isothermal surfaces*.

When the attraction of (positive or negative) matter, as for instance electricity, spread over a surface is considered, the density of the matter at any point is measured by the quantity of matter on an element of the surface, divided by that element.

The principal object of this paper is to prove the following theorems.

If upon E, one of the surfaces of equilibrium enclosing an attracting mass, its matter be distributed in such a manner that its density at any point P is equal to the attraction of M on P; then,

1. The attraction of the matter spread over E, on an external point, is equal to the attraction of M on the same point multiplied by 4π.

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* Translations of this paper have been published in Taylor's Scientific Memoirs for April 1842, and in the Nos. of Liouville's Journal for July and August 1842.

† See vol. III. p. 73 of this Journal.

2. The attraction of the matter on E, on an internal point, is nothing.

These theorems were proved in a previous paper, (see vol. III. p. 75,) from considerations relative to the uniform motion of heat; but in the following they are proved by direct integration.

Let u be the potential of M, on the point P, (xyz) in E. The components of the attraction of M on P, in the directions of x, y, z, are

$$-\frac{du}{dx},\ -\frac{du}{dy},\ -\frac{du}{dz};$$

and hence, if α, β, γ be the angles which a normal to E at P makes with these directions, the total attraction on P is

$$-\left(\frac{du}{dx}\cos\alpha+\frac{du}{dy}\cos\beta+\frac{du}{dz}\cos\gamma\right)\text{ or }-\frac{du}{dn},$$

if dn be an element of the normal through P.

This is therefore the expression for the density at P, of the matter we have supposed to be spread over E. Let ds be an element of E at P; let v' be the potential of E, on a point P', $(x'y'z')$, either within or without E; and let Δ be the distance from P to P'. Then

$$v'=-\left\{\int\frac{\left(\frac{du}{dx}\cos\alpha+\frac{du}{dy}\cos\beta+\frac{du}{dz}\cos\gamma\right)ds}{\Delta}\right\}=-\left\{\int\frac{\frac{du}{dn}ds}{\Delta}\right\}\ldots(a),$$

the brackets enclosing the integrals denoting that the integrations are to be extended over the whole surface E. Now for ds, we may choose any one of the expressions,

$$ds=\frac{dy\,dz}{\cos\alpha},\quad ds=\frac{dx\,dz}{\cos\beta},\quad ds=\frac{dx\,dy}{\cos\gamma}.$$

Hence any integral of the form

$$\{\int(A\cos\alpha+B\cos\beta+C\cos\gamma)\,ds\}$$

may be transformed into the sum of the three integrals,

$$(\iint A\,dy\,dz),\quad(\iint B\,dx\,dz),\quad(\iint C\,dx\,dy),$$

by using the first, second, and third of the expressions for ds, in the first, second, and third terms of the integral respectively.

Hence, if $A=\frac{d\phi}{dx}\psi$, $B=\frac{d\phi}{dy}\psi$, $C=\frac{d\phi}{dz}\psi$,

$$\left(\int\frac{d\phi}{dn}\psi ds\right)\text{ or }\left\{\int\left(\frac{d\phi}{dx}\cos\alpha+\frac{d\phi}{dy}\cos\beta+\frac{d\phi}{dz}\cos\gamma\right)\psi ds\right\},$$

$$=\left\{\iint\psi\left(\frac{d\phi}{dx}dy\,dz+\frac{d\phi}{dy}dx\,dz+\frac{d\phi}{dz}dx\,dy\right)\right\}\ldots(b),$$

the limits of the integrations relative to y and z, x and z, x and y, being so chosen as to include the whole of the surface considered.

Making use of this transformation in (a) we have

$$v' = -\left\{\iint\left(\frac{du}{dx}\,\frac{dydz}{\Delta} + \frac{du}{dy}\,\frac{dxdz}{\Delta} + \frac{du}{dz}\,\frac{dxdy}{\Delta}\right)\right\}\ldots(a').$$

Now $$\iint\frac{du}{dx}\,\frac{dydz}{\Delta} = \int\int dydz\int dx\left(\frac{d^2u}{dx^2}\,\frac{1}{\Delta} + \frac{du}{dx}\,\frac{d}{dx}\,\frac{1}{\Delta}\right)$$
$$= \int\int\int dxdydz\left(\frac{d^2u}{dx^2}\,\frac{1}{\Delta} + \frac{du}{dx}\,\frac{d}{dx}\,\frac{1}{\Delta}\right).$$

Hence, if the integrals in the second member include every point in the space contained between E, and another surface of equilibrium, E', without E, and which we shall suppose to be also without P', we have

$$\left\{\iint\frac{du}{dx}\frac{dydz}{\Delta}\right\}' - \left\{\iint\frac{du}{dx}\frac{dydz}{\Delta}\right\} = \iiint\left(\frac{d^2u}{dx^2}\frac{1}{\Delta} + \frac{du}{dx}\frac{d}{dx}\frac{1}{\Delta}\right)dxdydz,$$

the accent denoting that, in the term accented, the integrals are to be extended over the surface E'. Modifying in a similar manner the second and third terms of v', we have

$$\left\{\int\frac{\frac{du}{dn}\,ds}{\Delta}\right\}' - \left\{\int\frac{\frac{du}{dn}\,ds}{\Delta}\right\} \text{ or } \left\{\int\frac{\frac{du}{dn}\,ds}{\Delta}\right\}' + v'$$

$$= \iiint\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + \frac{du}{dx}\,\frac{d}{dx}\,.\,\frac{1}{\Delta} + \frac{du}{dy}\,\frac{d}{dy}\,.\,\frac{1}{\Delta} + \frac{du}{dz}\,\frac{d}{dz}\,.\,\frac{1}{\Delta}\right)dxdydz \quad \ldots.(c).$$

Now, for all points without M,

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0,$$

by a known theorem; and such points only are included in the integrals in the second member of (c).

Also, by integration by parts,

$$\iiint\frac{du}{dx}\frac{d}{dx}\frac{1}{\Delta}\,dxdydz = \iint u\,\frac{d}{dx}\frac{1}{\Delta}\,dydz - \iiint u\,\frac{d^2}{dx^2}\frac{1}{\Delta}\,dxdydz$$

$$= \left\{\iint u\,\frac{d}{dx}\frac{1}{\Delta}\,dydz\right\}' - \left\{\iint u\,\frac{d}{dx}\frac{1}{\Delta}\,dydz\right\} - \iiint u\,\frac{d^2}{dx^2}\frac{1}{\Delta}\,dxdydz.$$

Modifying similarly the two remaining terms of the second member of (c), we have

$$\left\{\int\frac{\frac{du}{dn}ds}{\Delta}\right\}' + v' = \left\{\iint u\left(\frac{d}{dx}\frac{1}{\Delta}dydz + \frac{d}{dy}\frac{1}{\Delta}dxdz + \frac{d}{dz}\frac{1}{\Delta}dxdy\right)\right\}'$$
$$- \left\{\iint u\left(\frac{d}{dx}\frac{1}{\Delta}\,dydz + \frac{d}{dy}\frac{1}{\Delta}\,dxdz + \frac{d}{dz}\frac{1}{\Delta}\,dxdy\right)\right\}$$
$$- \iiint u\left(\frac{d^2}{dx^2}\frac{1}{\Delta} + \frac{d^2}{dy^2}\frac{1}{\Delta} + \frac{d^2}{dz^2}\frac{1}{\Delta}\right) dxdydz \ldots.(c').$$

Now, since E and E' are surfaces of equilibrium, u is constant for each. Again,

$$\frac{d^2}{dx^2}\frac{1}{\Delta} + \frac{d^2}{dy^2}\frac{1}{\Delta} + \frac{d^2}{dz^2}\frac{1}{\Delta} = 0,$$

except when P coincides with P', at which point u has the value u'. Hence, the value of the integrals,

$$\iiint u\left(\frac{d^2}{dx^2}\frac{1}{\Delta} + \frac{d^2}{dy^2}\frac{1}{\Delta} + \frac{d^2}{dz^2}\frac{1}{\Delta}\right) dxdydz$$

is only affected by these elements, for which $u = u'$, and hence u may be taken without the integral sign, as being constant and equal to u'. If therefore, for brevity, we put

$$\iint\left(\frac{d}{dx}\frac{1}{\Delta}\,dydz + \frac{d}{dy}\frac{1}{\Delta}\,dxdz + \frac{d}{dz}\frac{1}{\Delta}\,dxdy\right) = (h) \text{ or } (h)' \ldots.(d),$$

according as the integrals refer to E, or to E',

$$\text{and} \iiint\left(\frac{d^2}{dx^2}\frac{1}{\Delta} + \frac{d^2}{dy^2}\frac{1}{\Delta} + \frac{d^2}{dz^2}\frac{1}{\Delta}\right)' dxdydz = k \ldots\ldots.(e),$$

the integrations including every point between E and E'; equation (c') becomes

$$\left\{\int\frac{\frac{du}{dn}ds}{\Delta}\right\}' + v' = (u)'(h)' - (u)(h) - u'k \ldots\ldots (c'').$$

Now it is obvious that, at a great distance from M, the surfaces of equilibrium are very nearly spherical. Let E' be taken so far off that it may be considered as spherical, without sensible error, and let γ be the distance of any point in E', from the centre, a fixed point in M, or, which is the same, the radius of the sphere. Then $-\frac{du}{dn}$, or $-\frac{du}{d\gamma}$ is the attraction of M, on a point in E', and is therefore equal to $\frac{M}{\gamma^2}$,

and therefore, by the known expression for the potential of a uniform spherical shell, on an interior point,

$$-\left\{\int\frac{\frac{du}{dn}\,ds}{\Delta}\right\}', \text{ or } \frac{M}{\gamma^2}\left\{\int\frac{ds}{\Delta}\right\}' = \frac{M}{\gamma^2}\,4\pi\gamma = 4\pi\,(u)' \ldots\ldots (f).$$

It now only remains to determine the integrals (h), $(h)'$, and k.

By putting, in (b), $\psi = 1$, $\phi = \frac{1}{\Delta}$, we find the following transformation, for (h),

$$h = \int\frac{d\,\frac{1}{\Delta}}{du}\,ds = -\int\frac{d\Delta}{du}\,\frac{ds}{\Delta^2}.$$

Now let the point (xyz) be referred to the polar co-ordinates, γ, θ, ϕ. Then, if P' be pole, $\gamma = \Delta$. Also, if ψ be the angle between Δ and dn, the expression for ds is

$$ds = \frac{\Delta^2 \sin\theta d\theta d\phi}{\cos\psi}, \quad \text{or, since } \cos\psi = \frac{d\Delta}{dn},$$

$$ds = \frac{\Delta^2\sin\theta d\theta d\phi}{\frac{d\Delta}{dn}}.$$

Hence, $\qquad h = -\iint \sin\theta d\theta d\phi.$

If P' be within the surface to which the integrals refer, the limits for θ are 0 and π, and for ϕ, 0 and 2π, and in that case, $h = -4\pi$; therefore, since P' is always within E',

$$(h)' = -4\pi \ldots\ldots\ldots (g).$$

If P' be without the surface considered, then, for each value of θ, we must take the sum of the expressions

$$-\sin\theta d\theta d\phi, \text{ and } -\sin\theta\,(-d\theta)\,d\phi,$$

and therefore, each element of the integral is destroyed by another equal to it, but with a contrary sign, and the value of the complete integral is therefore zero.

Hence, according as P' is without or within E,

$$(h) = 0, \text{ or } (h) = -4\pi \ldots\ldots\ldots (h).$$

Again, to find the value of k, we have, by dividing it into three terms, and integrating each once,

$$k = \left\{\iint\left(\frac{d}{dx}\,\frac{1}{\Delta}\,dy\,dz + \frac{d}{dy}\,\frac{1}{\Delta}\,dx\,dz + \frac{d}{dz}\,\frac{1}{\Delta}\,dx\,dy\right)\right\}$$
$$-\left\{\iint\left(\frac{d}{dx}\,\frac{1}{\Delta}\,dy\,dz + \frac{d}{dy}\,\frac{1}{\Delta}\,dx\,dz + \frac{d}{dz}\,\frac{1}{\Delta}\,dx\,dy\right)\right\}$$
$$= (h)' - (h) = -4\pi - 0, \quad \text{or } = -4\pi + 4\pi;$$

and therefore, according as P' is without or within E,

$$k = -4\pi, \quad \text{or} \quad k = 0 \ldots\ldots\ldots (k).$$

Hence, making use of (f), (g), (h), (k), in (c''), we have

$$v' = 4\pi u', \quad \text{when } P' \text{ is without } E \ldots\ldots (1),$$

$$v' = 4\pi (u), \quad \text{when } P' \text{ is within } E \ldots\ldots (2).$$

From the first of these equations it follows, that the attraction of E, on a point without it, is the same as that of M, multiplied by 4π; and since the second shows that the potential of E, on internal points, is constant, we infer that the attraction of E on internal points is nothing.

These theorems, along with some others which were also proved in the previous paper in this Journal, already referred to, had, I have since found, been given previously by Gauss. One of the most important of these is the following. If a mass M be wholly within, or wholly without a surface, an equal mass may be distributed over this surface in such a manner that its attraction, in the former case on external points, and in the latter on internal, will be equal to the attraction of M, on the same points. This theorem, which was proved from physical considerations in the paper *On the Uniform Motion of Heat, &c.*, is proved analytically in Gauss' *Mémoire*, but the same method is used in both to infer from it the truth of propositions (1) and (2).

From Prop. (2) it follows that, if E be the surface of an electrified conducting body, the intensity of the electricity at any point will be proportional to the attraction of M on the point. Hence we have the means of finding an infinite number of forms for conducting bodies, on which the distribution of electricity can be determined.

Thus, if M consists of a group of material points, m_1, m_2, &c., whose co-ordinates are x_1, y_1, z_1; x_2, y_2, z_2, &c., the general equation to the surfaces of equilibrium is

$$\frac{m_1}{\{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2\}^{\frac{1}{2}}} + \frac{m_2}{\{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2\}^{\frac{1}{2}}} + \&c. = \lambda,$$

and the intensity of electricity at any point of a solid body, bounded by one of them, will be the value of

$$\left\{\left(\frac{d\lambda}{dx}\right)^2 + \left(\frac{d\lambda}{dy}\right)^2 + \left(\frac{d\lambda}{dz}\right)^2\right\}^{\frac{1}{2}},$$

at the point.

To take a simple case: Let there be only two material points, of equal intensity. The surface will then be a surface of revolution, and will be symmetrical with regard to a plane

perpendicular, through its point of bisection, to the line joining the two points, and would probably very easily be constructed, in practice. We should thus have a simple method of verifying numerically the mathematical theory of electricity. P. Q. R.

(*To be continued.*)

X.—NEW PROPERTY OF THE ELLIPSE AND HYPERBOLA.*

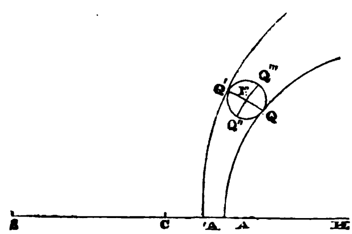
If a body, setting out from a given point, move so that the *difference* of its distances from two fixed points is always *greater*, or always *less*, than if it had moved over an equal space in any other way, its path will be an *ellipse*, of which the two fixed points are the foci. If it move so that the *sum* of its distances from the fixed points has that property, its path will be an *hyperbola*, of which the fixed points are the foci.

Taking the first case, let S, H, be the fixed points: then, since the difference of the distances is *always* to be the greatest or least, it must be true for every point of the path; and consequently the body, in moving from a position P over an elementary space PQ of given length, must move in such a direction as to satisfy the condition $SQ - HQ$, a maximum or a minimum.

Now, with the centre P and radius PQ, describe a circle; then, in what direction soever the body move, it must be found somewhere in the circumference of this circle when it has moved over a space equal to PQ: we have therefore to enquire, what point Q in the circumference satisfies the condition

$$SQ - HQ,$$

a maximum or a minimum. With foci S, H, describe two hyperbolas: one of which,



AQ, touches the circle on the side towards H; and the other, $A'Q'$, on the side towards S. Let Q, Q', be the points of contact; and bisect SH in C. Then an hyperbola, having the foci S, H, and a major axis greater than $2CA$, would fall within AQ, and therefore would not reach the circle; AQ is therefore the hyperbola of greatest major axis which can have

* From a Correspondent.

a point in common with the circle: consequently, $SQ - HQ$ is greater than if Q were situated in any other point of the circle; and therefore Q is the point to which the body must move, in order that the difference of its distances from S and H may always be greater than if it had moved through the same space in another direction. But PQ is a normal to the circle, and therefore to the hyperbola; and consequently SQ, HQ, are inclined at equal angles to PQ, which is an element of the required path. Hence the body moves in a curve, the tangent at every point P of which is inclined at equal angles to SP, HP; the curve is therefore an ellipse, of which S, H, are the foci. By the aid of the hyperbola $A'Q'$, we may shew that the body must move in the same ellipse to Q', if the difference of its distances from S and H is always to be the *least* possible. Hence it appears, that when a body moves in an ellipse, the difference of its focal distances is always a maximum or always a minimum, according as it is approaching towards or receding from the major axis.

If with the foci S and H we had described ellipses touching the circle in Q'' and Q''', we might have shewn that the body must move from P to Q'', or from P to Q''', in order that the *sum* of its distances from S and H may always be the least or the greatest possible. Hence $Q''PQ'''$ is ultimately an arc of an hyperbola, of which S and H are the foci. And hence, when a body moves in an hyperbola, the sum of its focal distances is always a maximum or always a minimum, according as it is receding from or approaching towards the major axis.

V.

XI.—MATHEMATICAL NOTES.

1. *Note on Mr. Bronwin's Paper on Elliptic Integrals.*—Jacobi's formulæ (8), (13), in p. 38, and the second formula in p. 37, of the *Nova Fundamenta, &c.* which Mr. Bronwin objects to in the case of m even, are perfectly correct. His own do certainly fail in that case, and the reason is obvious enough. The formulæ of Jacobi in question, adapted to the notation of the paper referred to, are

$$s.a.v = \frac{s.a.u\; s.a\,(u+2\omega)\ldots s.a\,\{u+2\,(n-1)\,\omega\}}{s.a\,(K-2\omega)\,s.a\,(K-4\omega)\ldots s.a\,\{K-2(n-1)\omega\}} \ldots . (1),$$

$$c.a.v = \frac{c.a.u.c.a\;(u+2\omega)\;\ldots\;c.a\;\{u+2\;(n-1)\;\omega\}}{c.a.2\omega.c.a.4\omega\;\ldots\;c.a\;(2n-2)\;\omega} \;\ldots\ldots\; (2),$$

$$\frac{1}{\beta} = \frac{s.a\;(K-2\omega)\;s.a\;(K-4\omega)\;\ldots\;s.a\;\{K-2\,(n-1)\;\omega\}}{s.a.2\omega\;s.a.4\omega\;\ldots\;s.a.2\,(n-1)\,\omega} \;\ldots (3).$$

The second of which coincides with Mr. Bronwin's, while he has for the denominator of the first and third,

$$s\,a.\omega\ s.a.3\omega\ \ldots\ldots\ s.a\,(2n-1)\,\omega,$$

a quantity which, as he remarks, vanishes when m is even, (and m'; however the passage refers particularly to $m'=0$). Jacobi's denominator does not vanish on the same supposition. Assume that from the equation (2) we may deduce one of the form m (1), only having a constant C for its denominator. In the equation (2), let u have the value ω assigned to it. $c.a.v$ contains the factor $c.a.n\omega$, which for

$$\omega=\frac{(2r+1)\,K+2r'K'\sqrt{(-1)}}{n}$$

vanishes, while for

$$\omega=\frac{2rK+2r'K'\sqrt{(-1)}}{n},\quad \omega=\frac{2rK+(2r'+1)\,K'\sqrt{(-1)}}{n'},$$

$$\omega=\frac{(2r+1)\,K+(2r'+1)\,K'\sqrt{(-1)}}{n},$$

it reduces itself to $(-1)^{r+r'}$, ∞, $\dfrac{K'\sqrt{(-1)}}{n}$, respectively. In the first case the corresponding value of $s.a.v$ is of course ± 1, and we have therefore the equation

$$\pm C=s.a.\omega\ s.a.3\omega\ \ldots\ldots\ s.a\,(2n-1)\,\omega,$$

Mr. Bronwin's divisor, which is therefore equal to Jacobi's in this particular case only. The two next cases give no results, and the last gives

$$\pm\frac{1}{k}\,C=s.a.\omega.s.a.3\omega\ \ldots\ldots\ s.a\,(2n-1)\,\omega\,;$$

or in this case the divisor is

$$k\,.\,s.a.\omega.s.a.3\omega\ \ldots\ldots\ s.a\,(2n-1)\,\omega.$$

C.

2. *Solution of a Geometrical Problem.*—The sum of the squares of the perpendiculars let fall from n given points on a plane is constant; the plane envelopes a central surface of the second order, having its centre at the centre of gravity of the n points, and its axes coincident with the principal axes of the system of n points.

Assuming any rectangular axes, let $(x'y'z')$, $(x''y''z'')$ ...... $(x^{\prime}y^{\prime}z^{\prime})$, be the projective co-ordinates of the n points, ξ, v, ζ, the tangential co-ordinates of the plane, the sum of the squares $= nk^2$; then we shall have

$$P_{\prime}^2+P_{\prime\prime}^2+P_{\prime\prime\prime}^2\ \ldots\ldots\ P_n^2=nk^2,$$

or as $P_{,} = \dfrac{(x'\xi + y'\upsilon + z'\zeta - 1)}{\sqrt{(\xi^2 + \upsilon^2 + \zeta^2)}}$, $P_{,,} = \dfrac{(x''\xi + y''\upsilon + z''\zeta - 1)}{\sqrt{(\xi^2 + \upsilon^2 + \zeta^2)}}$, &c.

Squaring and adding,

$$\{x'^2 + x''^2 + x'''^2 \ldots\}\,\xi^2 + \{y'^2 + y''^2 + y'''^2 \ldots\}\,\upsilon^2 + \{z'^2 + z''^2 + z'''^2 \ldots\}\,\zeta^2$$
$$+\, 2\,\{x'y' + x''y'' + x'''y'''\}\,\xi\upsilon + 2\,\{x'z' + x''z'' + x'''z'''\}\,\xi\zeta$$
$$+\, 2\,\{y'z' + y''z'' + y'''z''' \ldots\}\,\upsilon\zeta + 2\,\{x'z' + x''z'' + x'''z''' \ldots\}\,\xi\zeta$$
$$-\, 2\,\{x + x' + x'' \ldots\}\,\xi - 2\,\{y' + y'' + y''' \ldots\}\,\upsilon - 2\,\{z' + z'' + z''' + z^{iv}\}\,\zeta + n$$
$$= nk^2\,\{\xi^2 + \upsilon^2 + \zeta^2\};$$

which is the tangential equation of a central surface of the second order (when the absolute term is unity in an equation of this nature, the semi-coefficients of the linear terms ξ, υ, ζ, are co-ordinates of the centre); hence the co-ordinates of the centre of the surface are

$$\frac{x' + x'' + x''' \ldots}{n}, \quad \frac{y' + y'' + y''' \ldots}{n}, \quad \frac{z' + z'' + z''' \ldots}{n};$$

but these are the co-ordinates of the centre of gravity of the n points: hence, let the origin of co-ordinates be supposed to have been originally placed at the centre of gravity of the points, and the axes of co-ordinates coinciding with the principal axes of the system of n points; and let a, b, c, denote the radii of gyration of the system round the axes of X, Y, Z, respectively.

Then we shall have the following equations:

$$x'^2 + x''^2 + x'''^2 \ldots = na^2,\; y'^2 + y''^2 + y'''^2 \ldots = nb^2,\; z'^2 + z''^2 + z'''^2 \ldots = nc^2,$$
$$x'y' + x''y'' + x'''y''' = 0,\; x'z' + x''z'' + x'''z''' \ldots = 0,\; y'z' + y''z'' + y'''z''' = 0,$$
$$x' + x'' + x''' \ldots = 0,\;\; y' + y'' + y''' + y^{iv} = 0,\;\; z' + z'' + z''' + z^{iv} = 0:$$

making these substitutions in the original equation, and dividing by n, we shall have

$$(k^2 - a^2)\,\xi^2 + (k^2 - b^2)\,\upsilon^2 + (k^2 - c^2)\,\xi^2 = 1,$$

the tangential equation of a central surface of the second order, the squares of whose semiaxes are $(k^2 - a^2)$, $(k^2 - b^2)$, and $(k^2 - c^2)$.

The distances of the foci of the principal sections of this surface from the centre are independent of k; hence, if different groups of perpendiculars are let fall from the same n points on so many different planes, these planes will envelope as many confocal surfaces of the second order.

J. B.

3. *Note on the Measure of Intensity in the Theory of Light.*—The reason assigned by Mr. Airy (*Tracts*, p. 296, note,) for taking the square of the coefficient of the disturbance as the measure of the intensity of light, appears to be not very satisfactory: the following considerations may perhaps place the matter in a clearer light. They are taken from a paper by Abria in *Liouville's Journal*, tom. IV. p. 248.

The mechanical effect of a body in motion is measured by the vis viva. Now, in order that light may produce a sensible impression on the retina, it is necessary that the action should continue for a time which, though short, is yet very much greater than the time of vibration of a particle of the luminiferous ether: therefore the measure of the effect must be the aggregate vis viva during the time necessary to produce sensation. But it is generally required only to find the ratio of the intensities at two different points; and hence it is sufficient to calculate the ratio between the sums of the squares of the velocities of the molecules at the two points during the time of one vibration: for in consequence of the great number of vibrations which take place before a sensible effect is produced, we may suppose that number to be an integer, and the same for the two points.

Now, let the disturbance be represented by

$$a \sin \frac{2\pi}{\lambda}(vt - x);$$

then the velocity of the molecule will be

$$\frac{2\pi v}{\lambda} a \cos \frac{2\pi}{\lambda}(vt - x);$$

and if τ be the time of a vibration, the sum of the vis viva during that time will be represented by

$$\int_0^\tau dt\, \frac{4\pi^2 v^2}{\lambda^2} a^2 \cos^2 \frac{2\pi}{\lambda}(vt - x).$$

But $\tau = \frac{\lambda}{v}$; therefore, integrating between the limits, we find as the measure of the effect,

$$2\pi^2 \frac{v}{\lambda} a^2,$$

which is proportional to the square of the coefficient.

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I.—PROPOSITIONS IN THE THEORY OF ATTRACTION.

PART II.

I SHALL now prove a general theorem, which comprehends the propositions demonstrated in Part I., along with several others of importance in the theories of electricity and heat.

Let M and M_1 be two bodies, or groups of attracting or repelling points; and let v and v_1 be their potentials on xyz; let R and R_1 be their total attractions on the same point; and let θ be the angle between the directions of R and R_1, and $\alpha\beta\gamma$, $\alpha_1\beta_1\gamma_1$, the angles which they make with xyz. Let S be a closed surface, ds an element, corresponding to the co-ordinates xyz; and P and P_1 the components of R, R_1, in a direction perpendicular to the surface at ds. Then we have

$$R\cos\alpha = -\frac{dv}{dx}, \quad R\cos\beta = -\frac{dv}{dy}, \quad R\cos\gamma = -\frac{dv}{dz},$$

$$R_1\cos\alpha_1 = -\frac{dv_1}{dx}, \quad R_1\cos\beta_1 = -\frac{dv_1}{dy}, \quad R_1\cos\gamma_1 = -\frac{dv_1}{dz},$$

$$\cos\theta = \cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1;$$

hence, $$\frac{dv}{dx}\frac{dv_1}{dx} + \frac{dv}{dy}\frac{dv_1}{dy} + \frac{dv}{dz}\frac{dv_1}{dz} = RR_1\cos\theta.$$

Hence, $\iiint RR_1\cos\theta\, dx\, dy\, dz$

$$= \iiint\left(\frac{dv}{dx}\frac{dv_1}{dx} + \frac{dv}{dy}\frac{dv_1}{dy} + \frac{dv}{dz}\frac{dv_1}{dz}\right)dx\, dy\, dz \ \ldots\ldots (a),$$

where we shall suppose the integrals to include every point in the interior of S. Now, by integration by parts, the second member may be put under the form,

X

$$\iint v_1 \left(\frac{dv}{dx}\,dy\,dz + \frac{dv}{dy}\,dx\,dz + \frac{dv}{dz}\,dx\,dy\right)$$
$$- \iiint v_1' \left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) dx\,dy\,dz \ldots (b);$$

where the double integrals are extended over the surface S, and the triple integrals as before, over every point in its interior. If we transform the first term of this by (b), Part I., and observe that $-\frac{dv}{dn} = P$, it becomes

$$- \iint v_1\, P ds.$$

Again, $$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 0 \ldots\ldots (c).$$

except when xyz is a point of the attracting mass.

If this be the case, and if k be the density of the matter at the point, we have

$$\left.\begin{aligned} \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 4\pi k = 0 \\ \text{therefore } \left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) dx\,dy\,dz + 4\pi dm = 0 \end{aligned}\right\} \ldots\ldots (d).$$

Hence (a) is transformed into

$$\iiint RR_1 \cos\theta\, dx\,dy\,dz = 4\pi \iiint v_1\, dm - \iint v_1\, Pds \ldots\ (3);$$

similarly, by performing the integration in (a), on the terms

$$\frac{dv}{dx},\ \frac{dv}{dy},\ \frac{dv}{dz},\ \text{instead of } \frac{dv_1}{dx},\ \frac{dv_1}{dy},\ \frac{dv_1}{dz},$$

we should have found

$$\iiint RR_1 \cos\theta\, dx\,dy\,dz = 4\pi \iiint v dm_1 - \iint vP_1\, ds \ldots\ (4).$$

If the triple integrals in (a) were extended over all the space without S, or over every point between S, and another surface, S', enclosing it, at an infinite distance, it may be shown, as in part I., that the superior values of the double integrals in (b), corresponding to S', vanish. Hence, the inferior values being those which correspond to S, we have, instead of (3) and (4),

$$\iiint RR_1 \cos\theta\, dx\,dy\,dz = 4\pi \iiint v_1\, dm + \iint v_1\, Pds \ldots\ldots (5),$$
$$\iiint RR_1 \cos\theta\, dx\,dy\,dz = 4\pi \iiint v\, dm_1 + \iint v\, P_1 ds \ldots\ldots (6).$$

It is obvious that v and v_1 in these equations may be any functions, each of which satisfy equations (c) and (d), whether we consider them as potentials or temperatures, or as mere analytical functions with the restriction that, in (5) and (6), v

and v_1 must be such as to make $\iint v_1 P ds$ and $\iint v P_1 ds$ vanish at S'. If each of them satisfy (c) for all the points within the limits of the triple integrals considered, dm and dm_1 will each vanish; but if there be any points within the limits, for which either v or v_1 does not satisfy (c), the value of dm or dm_1 at those points will be found from (d).

Thus let $v_1 = 1$, for every point. Then we must have $dm_1 = 0$. Also $R_1 = 0$, $P_1 = 0$.

Hence, (3) becomes

$$\iint P ds = 4\pi \iiint dm = 4\pi m \dots\dots\dots\dots (7),$$

if m be the part of M within S. This expression is independent of the quantity of matter without S, and if $m = 0$, it becomes

$$\iint P ds = 0 \dots\dots\dots\dots\dots\dots (8).$$

If M be a group of sources of heat, in a solid body, P will be the flux across a unit of surface, at the point xyz. Hence the total flux of heat across S is equal to the sum of the expenditures from all the sources in the interior; and if there be no sources in the interior, the whole flux is nothing. Both these results, though our physical ideas of heat would readily lead us to anticipate them, are by no means axiomatic when considered analytically. In exactly a similar manner, Poisson* proves that the total flux of heat out of a body during an instant of time is equal to the sum of the diminutions of heat of each particle of the body, during the same time. This follows at once from (7). For, if we suppose there to be no sources of heat within S, but the temperature of interior points to vary with the time, on account of a non-uniform initial distribution of heat, we have

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = \frac{dv}{dt}.$$

Hence, by (d), we must use $-\dfrac{dv}{dt}\,dx\,dy\,dz$, instead of $4\pi dm$ and therefore (7) becomes

$$\iint P ds = \iiint \frac{-dv}{dt}\,dx\,dy\,dz.$$

It was the analysis used by Poisson, in the demonstration of this theorem, that suggested the demonstrations given in part I., of propositions (1) and (2).

As another example of the application of the theorem expressed by (3) and (4), let v_1 be the potential of a unit of mass, concentrated at a fixed point, $x'y'z'$. Hence, $M_1 = 1$, and

* See *Théorie de la Chaleur*, p. 177.

$dm_1 = 0$, except when xyz, at which dm_1 is supposed to be situated, coincides with $x'y'z'$; and, if Δ be the distance of xyz from $x'y'z'$, $v_1 = \dfrac{1}{\Delta}$.

Hence, according as $x'y'z'$ is without or within S,

$$\iiint v dm_1 = 0, \quad \text{or} \iint v dm_1 = v' \iiint dm_1 = v' \text{ ...... } (e),$$

the triple integrals being extended over the space within S. Now let us suppose M to be such, that v has a constant value (v) at S. Then $\iint vP_1 ds = (v)\iint P_1 ds$, which by (7), is $= 0$, or to $4\pi(v)$, according as $x'y'z'$ is without or within S. Hence, by comparing (3) and (4) we have, in the two cases,

$$4\pi \iiint \frac{dm}{\Delta} - \iint \frac{Pds}{\Delta} = 0, \text{ or } \iint \frac{Pds}{\Delta} = 4\pi \iiint \frac{dm}{\Delta} = 4\pi v' \text{...(9)},$$

$$\text{and } 4\pi \iiint \frac{dm}{\Delta} - \iint \frac{Pds}{\Delta} = -4\pi(v) + 4\pi v';$$

$$\text{therefore} \qquad \iint \frac{Pds}{\Delta} = 4\pi(v) \text{ .... } (10).$$

These are the two propositions (1) and (2) proved in Part I., which are therefore, as we see, particular cases of the general theorem expressed by (3) and (4).*

If $v = v_1$, and if both arise from sources situated without S, (3) becomes

$$\iiint R^2 \, dx \, dy \, dz = \iint vPds \text{ .... } (11),$$

a proposition given by Gauss. If v have a constant value (v), over S, we have

$$\iint vPds = (v)\iint Pds = 0, \quad \text{by (8)},$$

$$\text{hence} \qquad \iiint R^2 \, dx \, dy \, dz = 0.$$

Therefore $R = 0$, and $v = (v)$, for interior points. Hence, if the potential produced by any number of sources, have the same value over every point of a surface which contains none of them, it will have the same value for every interior point also. If we consider the sources to be spread over S, it follows that $v = (v)$, at the surface is a condition which implies that the attraction on an interior point will be nothing. Hence the

* It may be here proper to state that these theorems, which were first demonstrated by Gauss, are the subject of a Mémoire by M. Chasles, in the Additions to the *Connaissance des Temps* for 1845, published in June, 1842. In this Mémoire he refers to an announcement of them, without a demonstration, in the *Comptes Rendus des Séances de l'Académie des Sciences*, Feb. 11th, 1839, a date earlier than that of M. Gauss' Mémoire, which was read at the Royal Society of Gottingen, in March 1840.

sole condition for the distribution of electricity over a conducting surface, is that its attraction shall be every where perpendicular to the surface, a proposition which was proved from indirect considerations, relative to heat, in a former paper.*

In exactly a similar manner, if none of the sources be without S, by means of (5) and (7), it may be shown that

$$\iiint R^2\, dx\, dy\, dz = 4\pi M(v) \ldots\ldots\ldots\ldots (12);$$

the triple integrals being extended over all the space without S. Hence a quantity of matter μ can only be distributed in one way on S, so as to make (v) be constant. For if there were two distributions of μ, each making (v) constant, there would be a third, corresponding to their difference, which would also make (v) constant. The whole mass in the third case would be nothing. Hence, by (12), we must have $\iiint R^2\, dx\, dy\, dz = 0$, and therefore $R = 0$, for external points; and, since (v) is constant at the surface, R must be $= 0$, for interior points also. Now this cannot be the case unless the density at each point of the surface be nothing, on account of the theorem of Laplace, that, if ρ be the density at any point of a stratum which exerts no attraction on interior points, its attraction on an interior point, close to the surface, will be $4\pi\rho$. This important theorem, which shows that there is only one distribution of electricity on a body, that satisfies the condition of equilibrium, was first given by Gauss. It may be readily extended, as has been done by Liouville,† to the case of any number of electrified bodies, influencing one another, by supposing S to consist of a number of isolated portions, which will obviously not affect the truth of (5) and (6).

Then, if we suppose v to have the constant values, (v), $(v)'$, &c., at the different surfaces, and the quantities of matter on these surfaces to be M, M', &c. we should have, instead of (11),

$$\iiint R^2\, dx\, dy\, dz = 4\pi \left\{M(v) + M'(v)' + \&c.\right\} \ldots\ldots (13),$$

and from this it may shown as above, that there is only one distribution of the same quantities of matter, M, M', &c. which satisfies the conditions of equilibrium.

If both M and M_1 be wholly within S, by comparing (5) and (6), or if both be without S, by comparing (3) and (4), we have

$$\iint Pv_1 ds = \iint P_1 v ds \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

Now let S be a sphere, and let $r\theta\phi$ be the polar co-ordinates, from the centre as pole, of any point in the surface, to

* See Vol. III., p. 74.

† See Note to M. Chasles' Memoire in the *Connaissance des Temps*, for 1845.

which the potentials v and v_1 correspond. Then we shall have $P = -\frac{dv}{dr}$, $P_1 = -\frac{dv_1}{dr}$, and we may assume $ds = r^2 \sin\theta\, d\theta\, d\phi$. Hence (13) becomes

$$\int_0^\pi\int_0^{2\pi} v_1 \frac{dv}{dr} \sin\theta\, d\theta\, d\phi = \int_0^\pi\int_0^{2\pi} v\frac{dv_1}{dr} \sin\theta\, d\theta\, d\phi \ldots\ldots (15).$$

This equation leads at once to the fundamental property of Laplace's coefficients. For if v and v_1 be of the forms $Y_m r^m$, ${}_1Y_n r^n$, m and n being any positive or negative integers, zero included, and Y_m and ${}_1Y_n$ being independent of r, we have, by substitution in (15),

$$m\int_0^\pi\int_0^{2\pi} Y_m\, {}_1Y_n \sin\theta\, d\theta\, d\phi = n\int_0^\pi\int_0^{2\pi} Y_m\, {}_1Y_n \sin\theta\, d\theta\, d\phi.$$

If m be not $= n$, this cannot be satisfied, unless

$$\int_0^\pi\int_0^{2\pi} Y_m\, {}_1Y_n \sin\theta\, d\theta\, d\phi = 0 \ldots\ldots\ldots\ldots\ldots (16).$$

This is the fundamental property of Laplace's coefficients.

There are some other applications of the general theorem which has been established, especially to the Theory of Electricity, which must however be left for a future opportunity.

II.—ON THE LINEAR MOTION OF HEAT.

PART II.

Let us now endeavour to find the general form of ${}_0v$, for positive and negative values of x, which is producible by any distribution of heat, an infinite time previously, or which is the same, to find the form of the function f, which renders v, or $\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty} d\alpha\, \varepsilon^{-\alpha^2} f(x + 2\alpha t^{\frac{1}{2}})$, possible for all values of χ, and for all values of t, back to $-\infty$.

If v be possible for all values of t, it may be represented by

$$\Sigma\left(P_i \cos\frac{2i\pi t}{p} + Q_i \sin\frac{2i\pi t}{p}\right) \ldots\ldots\ldots (d),$$

where P_i and Q_i are functions of x, which it is our object to determine.

Modifying (a) and (b), so that the multiplier of $d\alpha\, \varepsilon^{-\alpha^2}$, in the first members, may be of the form $f(x + 2\alpha t^{\frac{1}{2}})$, and the second members of the forms $P\cos(2mt)$, $Q\sin(2mt)$, and putting $m = \frac{i\pi}{p}$, we have

$$\int_{-\infty}^{\infty} d\alpha\, \varepsilon^{-\alpha^2} \varepsilon^{-(x+2\alpha t^{\frac{1}{2}})\sqrt{\frac{i\pi}{p}}} \frac{\sin}{\cos}\left\{\sqrt{\frac{i\pi}{p}}\,(x+2\alpha t^{\frac{1}{2}})\right\}$$

$$= \pi^{\frac{1}{2}} \varepsilon^{-x\sqrt{\frac{i\pi}{p}}} \frac{\sin}{\cos}\left(x\sqrt{\frac{i\pi}{p}} - \frac{2i\pi t}{p}\right),$$

$$\int_{-\infty}^{\infty} d\alpha\, \varepsilon^{-\alpha^2} \varepsilon^{(x+2\alpha t^{\frac{1}{2}})\sqrt{\frac{i\pi}{p}}} \frac{\sin}{\cos}\left\{\sqrt{\frac{i\pi}{p}}\,(x+2\alpha t^{\frac{1}{2}})\right\}$$

$$= \pi^{\frac{1}{2}} \varepsilon^{x\sqrt{\frac{i\pi}{p}}} \frac{\sin}{\cos}\left(x\sqrt{\frac{i\pi}{p}} + \frac{2i\pi t}{p}\right).$$

Hence we see that the most general expression for v, when it is of the form (d), is

$$v = \sum_0^\infty \left[\varepsilon^{x\sqrt{\frac{i\pi}{p}}} \left\{ A_i \cos\left(x\sqrt{\frac{i\pi}{p}} + \frac{2i\pi t}{p}\right) + B_i \sin\left(x\sqrt{\frac{i\pi}{p}} + \frac{2i\pi t}{p}\right)\right\} \right.$$

$$\left. + \varepsilon^{-x\sqrt{\frac{i\pi}{p}}} \left\{ A_i' \cos\left(x\sqrt{\frac{i\pi}{p}} - \frac{2i\pi t}{p}\right) + B_i' \sin\left(x\sqrt{\frac{i\pi}{p}} - \frac{2i\pi t}{p}\right)\right\}\right] \ldots (12).$$

This then represents the most general state of the temperature of the body when the heat has been moving freely for an infinite time, and therefore, whatever be the initial distribution, the ultimate distribution must be of this form.

Putting $t = 0$, we find

$$_0v = \sum_0^\infty \left\{ \varepsilon^{x\sqrt{\frac{i\pi}{p}}} \left(A_i \cos x\sqrt{\frac{i\pi}{p}} + B_i \sin x\sqrt{\frac{i\pi}{p}}\right)\right.$$

$$\left. + \varepsilon^{-x\sqrt{\frac{i\pi}{p}}} \left(A_i' \cos x\sqrt{\frac{i\pi}{p}} - B_i' \sin x\sqrt{\frac{i\pi}{p}}\right)\right\} \ldots (13),$$

for the simplest form of the distribution at any period, which is producible after an indefinite time.

This expression consists of two independent parts, one containing $\varepsilon^{x\sqrt{\frac{i\pi}{p}}}$ as a factor in each term, and the other $\varepsilon^{-x\sqrt{\frac{i\pi}{p}}}$. By examining (12), we see that the former of these gives rise to a series of *waves* of heat, proceeding in the negative direction; and the latter to a series of waves proceeding in the positive direction; and that while a wave in the former system moves from $x = \infty$ to $x = -\infty$, and a wave in the latter from $x = -\infty$ to $x = \infty$, its *amplitude* diminishes from ∞ to 0. As the two systems of waves are precisely similar, we may confine our attention to one of them, the latter for instance, which consists of waves proceeding in the positive direction

The initial distribution which gives rise to them is

$$ {}_0v = \sum_0^\infty \varepsilon^{-x\sqrt{\frac{i\pi}{p}}} \left(A_i \cos x \sqrt{\frac{i\pi}{p}} - B_i \sin x \sqrt{\frac{i\pi}{p}} \right) \,..\, (e). $$

Now it has been already shown that the initial distribution $F(-x)$ on the negative side produces the same value of v_0. On comparing the expression for $F(-x)$, given by (8), with (e), we see that the positive part ${}_0v$ in the latter is ***turned over***, and added to the negative part, to make $F(-x)$. This should obviously make the values of v_0 be the same in the two cases; but the variable temperatures of every point not situated in the zero plane should be different. Hence we see how it is that the distribution, $F(-x)$, on the negative side, makes the temperature of the zero plane periodical, and therefore real for every value of t, and that of every other parallel plane, unperiodical, and impossible for negative values of t.

If, in (12), f_1t and f_2t be the parts of v_0 arising from the series, of which the coefficients are A_i, B_i, and A_i', B_i', the value of v becomes

$$ \left.\begin{aligned} v = \frac{1}{p} \sum_{-\infty}^{\infty} \Bigg[& \varepsilon^{x\sqrt{\frac{i\pi}{p}}} \int_{-\frac{1}{2}p}^{\frac{1}{2}p} dt' f_1t' \cos \left\{ x\sqrt{\frac{i\pi}{p}} + \frac{2i\pi t}{p}(t-t') \right\} \\ & + \varepsilon^{-x\sqrt{\frac{i\pi}{p}}} \int_{-\frac{1}{2}p}^{\frac{1}{2}p} dt' f_2t' \cos \left\{ x\sqrt{\frac{i\pi}{p}} - \frac{2i\pi t}{p}(t-t') \right\} \Bigg], \\ \text{or, when } p = \infty, \quad & \\ \pi v = \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} dt' \, [& \varepsilon^{\beta^{\frac{1}{2}}x} f_1t' \cos \{ \beta^{\frac{1}{2}}x + 2\beta(t-t') \} \\ & + \varepsilon^{-\beta^{\frac{1}{2}}x} f_2t' \cos \{ \beta^{\frac{1}{2}}x - 2\beta(t-t') \}]. \end{aligned}\right\} \,..\, (14). $$

From the latter of these forms, that given by Fourier (*Théorie de la Chaleur*, p. 544,) may be readily deduced, and by putting $f_1t' = 0$ in the former, we have the solution (given by Kelland, in his *Treatise on Heat*, p. 127,) which Fourier employed to express the diurnal and annual variations in the temperature of the earth at small depths. It is obviously suited to the case in which the temperature of the body, below the surface, is naturally constant, and all the periodical variations are produced by external causes, and proceed downwards, from the surface.

2. Let the body be supposed to be terminated by the zero plane, and to radiate heat across it, according to Newton's law; and let the external temperature be a given function, ξt, of the time. To find the state of the temperature of the

body, after any time has elapsed, the initial distribution in the body, or on the positive side of the zero plane, being ϕx.

Let the medium into which the surface radiates be supposed to be removed, and, instead of it, let the body extend infinitely on the negative side. The first thing to be done is to find the distribution on the negative side which will exactly supply the place of the radiation. The conditions which this must be chosen to satisfy, are

$$\left.\begin{array}{r} \left(\dfrac{dv}{dx}\right)_0 = h\,(v_0 - \xi t), \\ {}_0v = \phi x, \text{ when } x \text{ is positive}, \end{array}\right\} \ldots \text{(a)},$$

where h is the radiating power of the surface. If, in addition to the latter of these equations, we assume ψx to be the required initial distribution on the negative side, the variable temperature, v, of any point, will be given by (4).

Differentiating this equation, and putting $x = 0$ in the result, we have

$$\pi^{\frac{1}{2}}\left(\frac{dv}{dx}\right)_0 = \int_0^\infty da\,\varepsilon^{-a^2}\phi'(2at^{\frac{1}{2}}) + \int_{-\infty}^0 da\,\varepsilon^{-a^2}\psi'(2at^{\frac{1}{2}}) + \frac{\phi o - \psi o}{2t^{\frac{1}{2}}}.$$

Now ψo must be $= \phi o$, as otherwise, at the commencement of the variation, the radiation would be infinite. Hence we have, from (a),

$$\int_0^\infty da\,\varepsilon^{-a^2}\phi'(2at^{\frac{1}{2}}) + \int_{-\infty}^0 da\,\varepsilon^{-a^2}\psi'(2at^{\frac{1}{2}})$$

$$= h\left\{\int_0^\infty da\,\varepsilon^{-a^2}\phi(2at^{\frac{1}{2}}) + \int_{-\infty}^0 da\,\varepsilon^{-a^2}\psi(2at^{\frac{1}{2}}) - \pi^{\frac{1}{2}}\xi t\right\},$$

or $\displaystyle\int_0^\infty da\,\varepsilon^{-a^2}[\phi'(2at^{\frac{1}{2}}) + \psi'(-2at^{\frac{1}{2}}) - h\{\phi(2at^{\frac{1}{2}}) + \psi(-2at^{\frac{1}{2}})\}] = -\pi^{\frac{1}{2}}h\xi t.$

Hence, if

$$\xi t = \Sigma\left(A_i \cos\frac{2i\pi t}{p} + B_i \sin\frac{2i\pi t}{p}\right) \ldots \text{(b)},$$

and if Fx be determined by (8), then, using x instead of $2at^{\frac{1}{2}}$, we must have

$$\phi' x + \psi'(-x) - h\{\phi x + \psi(-x)\} = -hFx \;.\,.\; \text{(c)},$$

$$\text{or} \quad \frac{d\psi(-x)}{-dx} - h\psi(-x) = h(\phi x + Fx) - \frac{d\phi x}{dx};$$

$$\left.\begin{array}{l} \text{therefore } \psi(-x) = -\varepsilon^{-hx}\displaystyle\int \varepsilon^{hx}\left\{h(\phi x - Fx) - \frac{d\,\phi x}{dx}\right\}dx \\ \quad\text{or } \psi(-x) = \phi x - h\,\varepsilon^{-hx}\displaystyle\int \varepsilon^{hx}(2\phi x - Fx)\,dx, \end{array}\right\} \ldots (15).$$

The function ψ being determined from this, the solution of the problem is found by using the result in (4).

After the motion has continued for a long period of time, the irregularities of the initial distribution disappear, and the variations of the temperature of the body are reduced, by the periodical variations of the external temperature, to a permanently periodical state. Let us suppose that this permanent state has been reached when $t = 0$. That this may be the case, we must choose ϕx of such a form, that ψx when x is negative, and ϕx when x is positive, may make ${}_0v$ be of the form (e), No. 1. Let us therefore assume

$$\left.\begin{array}{l}\text{when } x \text{ is negative } \psi x = \\ \text{when } x \text{ is positive } \phi x = \end{array}\right\} \Sigma \varepsilon^{-x\sqrt{\frac{i\pi}{p}}}\left(a_i \cos x\sqrt{\frac{i\pi}{p}} - b_i \sin x\sqrt{\frac{i\pi}{p}}\right),$$

where a_i and b_i are to be determined so as to satisfy (c).

Using these values of ψx and ϕx in (c), and using for Fx its value (8), we have, by equating the coefficients of

$$(\varepsilon^{x\sqrt{\frac{i\pi}{p}}} + \varepsilon^{-x\sqrt{\frac{i\pi}{p}}}) \cos x\sqrt{\frac{i\pi}{p}}, \text{ and } (\varepsilon^{x\sqrt{\frac{i\pi}{p}}} - \varepsilon^{-x\sqrt{\frac{i\pi}{p}}}) \sin x\sqrt{\frac{i\pi}{p}},$$

in the two members of the resulting equation,

$$a_i\left(h + \sqrt{\frac{i\pi}{p}}\right) + b_i\sqrt{\frac{i\pi}{p}} = hA_i,$$

$$a_i\sqrt{\frac{i\pi}{p}} - b_i\left(\sqrt{\frac{i\pi}{p}} + h\right) = -hB_i;$$

$$\text{whence} \quad \left.\begin{array}{l} a_i = \dfrac{h\left\{A_i\left(\sqrt{\frac{i\pi}{p}} + h\right) - B_i\sqrt{\frac{i\pi}{p}}\right\}}{h^2 + 2h\sqrt{\frac{i\pi}{p}} + 2\frac{i\pi}{p}} \\[2em] b_i = \dfrac{h\left\{A_i\sqrt{\frac{i\pi}{p}} + B_i\left(\sqrt{\frac{i\pi}{p}} + h\right)\right\}}{h^2 + 2h\sqrt{\frac{i\pi}{p}} + 2\frac{i\pi}{p}} \end{array}\right\} \dots\dots (16).$$

If, in (12), we put A_i and B_i each equal to zero, and use these values of a_i and b_i instead of A_i' and B_i', the resulting expression is the variable temperature of any point. Let, for brevity,

$$D_i = \sqrt{\left(h^2 + 2h\sqrt{\frac{i\pi}{p}} + 2\frac{i\pi}{p}\right)},$$

$$\cos\delta_i = \frac{\sqrt{\frac{i\pi}{p}} + h}{D_i}, \qquad \sin\delta_i = \frac{\sqrt{\frac{i\pi}{p}}}{D_i}.$$

Hence, using for A_i and B_i, their values, which satisfy (*b*), we have

$$pv = h\sum_{-\infty}^{\infty} \varepsilon^{-x\sqrt{\frac{i\pi}{p}}} \int_{-\frac{1}{2}p}^{\frac{1}{2}p} dt' \xi t' \frac{\cos\left\{x\sqrt{\frac{i\pi}{p}} - \frac{2i\pi}{p}(t-t') + \delta_i\right\}}{D_i} \;.\;.\; (17),$$

which agrees with the expression given by Poisson, in p. 431 of his *Théorie de la Chaleur*.

N. N.

III.—ON THE INTERSECTION OF CURVES.

By Arthur Cayley, B.A. Fellow of Trinity College.

The following theorem is quoted in a note of Chasles' Aperçu Historique, &c. *Memoires de Bruxelles*, tom. XI. p. 149, where M. Chasles employs it in the demonstration of Pascal's theorem. "If a curve of the third order pass through eight of the points of intersection of two curves of the third order, it passes through the ninth point of intersection." The application in question is so elegant, that it deserves to be generally known. Consider a hexagon inscribed in a conic section. The aggregate of three alternate sides may be looked upon as forming a curve of the third order, and that of a remaining sides, a second curve of the same order. These two intersect in nine points, viz. the six angular points of the hexagon, and the three points which are the intersections of pairs of opposite sides. Suppose a curve of the third order passing through eight of these points, viz. the aggregate of the conic section passing through the angular points of the hexagon, and of the line forming two of the three intersections of pairs of opposite sides. This passes through the ninth point, by the theorem of Chasles, *i. e.* the three intersections of pairs of opposite sides lie in the same straight line, (since obviously the third intersection does *not* lie in the conic section), which is Pascal's theorem.

The demonstration of the above property of curves of the third order is one of extreme simplicity. Let $U = 0$, $V = 0$, be the equations of two curves of the third order, the curve of the same order which passes through eight of their points of intersection, (which may be considered as eight perfectly arbitrary points), and a ninth arbitrary point, will be perfectly determinate. Let U_0, V_0, be the values of U, V, when the co-ordinates of this last point are written in place of x, y. Then $UV_0 - U_0V = 0$, satisfies the above conditions, or it is the equation to the curve required; but it is an equation

which is satisfied by all the nine points of intersection of the two curves, *i. e.* any curve that passes through eight of these points of intersection, passes also through the ninth.

Consider generally two curves, $U_m = 0$, $V_n = 0$, of the orders m and n respectively, and a curve of the r^{th} order (r not less than m or n) passing through the mn points of intersection. The equation to such a curve will be of the form

$$U = u_{r-m} U_m + v_{r-n} V_n = 0,$$

u_{r-m}, v_{r-n}, denoting two polynomes of the orders $r - m$, $r - n$, with all their coefficients complete. It would at first sight appear that the curve $U = 0$ might be made to pass through as many as $\{1 + 2 \ldots + (r - m + 1)\} + \{1 + 2 \ldots (r - n + 1)\} - 1$, arbitrary points, *i. e.*

$$\tfrac{1}{2}(r - m + 1)(r - m + 2) + \tfrac{1}{2}(r - n + 1)(r - n + 2) - 1\,;$$

or, what is the same thing,

$$\tfrac{1}{2}r\,.(r + 3) - mn + \tfrac{1}{2}(r - m - n + 1)(r - m - n + 2)$$

arbitrary points, such being apparently the number of disposable constants. This is in fact the case as long as r is not greater than $m + n - 1$; but when r exceeds this, there arise, between the polynomes which multiply the disposable coefficients, certain linear relations, which cause them to group themselves into a smaller number of disposable quantities. Thus, if r be not less than $m + n$, forming different polynomes of the form $x^\alpha y^\beta . V_n$ [$\alpha + \beta =$ or $< m$], and multiplying by the coefficients of $x^\alpha y^\beta$ in U_m, and adding, we obtain a sum $U_m V_n$, which might have been obtained by taking the different polynomes of the form $x^\gamma y^\delta . U_m$ [$\gamma + \delta =$ or $< n$], multiplying by the coefficients of $x^\gamma y^\delta$ in V_n, and adding. Or we have a linear relation between the different polynomes of the forms $x^\alpha y^\beta V_n$, and $x^\gamma y^\delta U_m$. In the case where r is not less than $m + n + 1$, there are two more such relations, viz. those obtained in the same way from the different polynomes $x^\alpha y^\beta . x V_n$, $x^\gamma y^\delta . x U_m$, and $x^\alpha y^\beta . y V_n$, $x^\gamma y^\delta . y U_m$, &c.; and in general, whatever be the excess of r above $m + n + 1$, the number of these linear relations is

$$1 + 2 \ldots (r - m - n + 1) = \tfrac{1}{2}(r - m - n + 1)(r - m - n + 2).$$

Hence, if r be not less than $m + n$, the number of points through which a curve of the r^{th} order may be made to pass, in addition to the mn points which are the intersections of $U_m = 0$, $V_n = 0$, is simply $\tfrac{1}{2}r(r + 3) - mn$. In the case of $r = m + n - 1$, or $r = m + n - 2$, the two formulæ coincide. Hence we may enunciate the theorem—

"A curve of the r^{th} order, passing through the mn points of intersection of two curves of the m^{th} and n^{th} orders re-

spectively, may be made to pass through $\frac{1}{2}r(r+3)-mn+\frac{1}{2}(m+n-r-1)(m+n-r-2)$ arbitrary points, if r be not greater than $m+n-3$: if r be greater than this value, it may be made to pass through $\frac{1}{2}r(r+3)-mn$ points only."

Suppose r not greater than $m+n-3$, and a curve of the r^{th} order made to pass through

$$\tfrac{1}{2}r(r+3)-mn+\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

arbitrary points, and

$$mn-\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

of the mn points of intersection above. Such a curve passes through $\frac{1}{2}r(r+3)$ given points, and though the $mn-\frac{1}{2}(m+n-r-1)(m+n-r-2)$ latter points are not perfectly arbitrary, there appears to be no reason why the relation between the positions of these points should be such, as to prevent the curve from being *completely determined* by these conditions. But if it be so, it must pass through the remaining $\frac{1}{2}(m+n-r-1)(m+n-r-2)$ points of intersection, or we have the theorem—

"If a curve of the r^{th} order (r not less than m or n, not greater than $m+n-3$) pass through

$$mn-\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

of the points of intersection of two curves of the m^{th} and n^{th} orders respectively, it passes through the remaining

$$\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

points of intersection."

IV.—RESEARCHES IN ROTATORY MOTION.

By Andrew Bell.

1. This article contains some theorems in rotatory motion, respecting the effect of the centrifugal force arising from the rotation of a body about an axis, in producing rotation about another axis inclined at any angle to the former.

To avoid unnecessary circumlocution, I propose that, instead of the expression—the effect of the centrifugal force arising from a rotation about an axis in producing rotation about another axis, this more concise one should be used, namely, the centrifugal effect for one axis about another axis.

2. The centrifugal effects of any rotations for two rectangular axes about the third co-ordinate rectangular axis, is equal to the similar effect for either of these axes, when the rotation about it is equal to the resultant of the rotations about them both.

Let (x'), (y'), (z'), denote the three rectangular axes; q, r, the respective rotations about the two latter; ω' their resultant rotation about an axis (z) in their plane; and θ the inclination of (z) and (z').

The sum of the centrifugal effects for the axes (y'), (z'), about (x'), is

$$= (q^2 + r^2)\ \Sigma y'z'\Delta m,$$

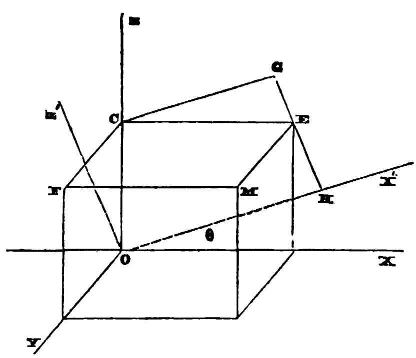
Δm denoting an element of the body.

But $q = \omega' \sin\theta$, and $r = \omega' \cos\theta$; and hence this effect becomes

$$= \omega'^2 \Sigma y'z'\Delta m;$$

which is the similar effect of an angular velocity ω' about either of the axes (y'), (z').

3. If to the second of two given axes* having any inclination, a third axis is drawn perpendicularly and in the same plane with the given axes; and if any given rotation about the first of the two given axes is decomposed into its constituents about the other two axes, then the centrifugal effect for the first axis about the second, is equal to the similar effect of the constituent rotation about the third axis.



Let Ox, Oy, Oz, be three rectangular axes, and Ox', Oz', perpendicular axes in the plane of (x) and (z); then if a rotation about (z) is decomposed into its constituents about (x') and (z'), the centrifugal effect for (z) about (x') is equal to that for (z') about (x').

If ω is the angular velocity about (z), then the centrifugal effect arising from this motion produces a force which may be represented in direction and quantity by CM, acting on an element Δm at M, and which can be resolved into CE and CF, so that the component forces parallel to the axes (x) and (y), are respectively

$$\omega^2 x\Delta m \text{ and } \omega^2 y\Delta m.$$

But CE may be resolved into CG parallel to (x'), and GE parallel to (z'), of which the former has no effect round (x');

* The axes concerned in any of these theorems are to be understood as co-originate, that is, as having the same origin.

and $GE = CE \sin\theta$, if θ is the inclination of (x) to (x'). Hence the forces acting on Δm that produce rotation about (x'), are,

$$CF \text{ or } EM \text{ parallel to } (y), \text{ and } = \omega^2 y\Delta m,$$

$$\text{and } GE \text{ parallel to } (z'), \text{ and } = \omega^2 \sin\theta x\Delta m.$$

The former of these forces acts at the extremity of an ordinate $z' = EH$, and the latter at that of an ordinate $y = EM$. Hence, as they conspire, the sum of their momenta about (x') is

$$= \omega^2\, \Sigma yz'\Delta m + \omega^2 \sin\theta\, \Sigma xy\Delta m.$$

But $x = x' \cos\theta - z' \sin\theta$, and hence

$$\omega^2 \sin\theta\, \Sigma xy\Delta m = \omega^2 \sin\theta \cos\theta\, \Sigma x'y\Delta m - \omega^2 \sin^2\theta\, \Sigma yz'\Delta m.$$

And since $\Sigma x'y\Delta m$ can have no effect about (x'), therefore the effect round (x') is

$$= \omega^2 (1 - \sin^2\theta)\, \Sigma yz'\Delta m = \omega^2 \cos^2\theta\, \Sigma yz'\Delta m.$$

Now if ω' is the constituent rotation about (z'), resulting from the decomposition of the rotation about (z) into its constituents about (z') and (x'); then the centrifugal effect for (z') about (x') is $\quad = \omega'^2\, \Sigma yz'\Delta m.$

And since $\omega' = \omega \cos\theta$, the latter expression becomes

$$= \omega^2 \cos^2\theta\, \Sigma yz'\Delta m,$$

which is the same as the above expression.

4. The effect of the centrifugal force for any axis about another, whatever be their inclination, is the same as its similar effect for an axis perpendicular to the first, and in the same plane with these two axes, reduced to its constituent effect about the second axis; considering these effects as measured by momenta, applied at a given distance from these axes.

Let Oz and Ox' be any two axes, and Ox another axis perpendicular to the first, and the rotation about Oz be denoted by ω.

The centrifugal effect for (z) about (x') was already proved to be (3) $\quad = \omega^2 \cos^2\theta\, \Sigma yz'\Delta m,$

ω being the angular velocity about (z). But the similar effect for (z) about (x) is

$$= \omega^2\, \Sigma yz\,\Delta m;$$

and this effect being decomposed into its constituents about (x') and (z'), gives for the former

$$\omega^2 \cos\theta\, \Sigma yz\,\Delta m.$$

And since $z = z' \cos\theta + x' \sin\theta$, therefore this constituent

$$= \omega^2 \cos^2\theta\, \Sigma yz'\Delta m + \omega^2 \sin\theta \cos\theta\, \Sigma yx'\Delta m;$$

and since $\Sigma yx'\Delta m$ can produce no effect round (x'), the latter expression becomes

$$\omega^2 \cos^2\theta\, \Delta yz'\Delta m,$$

which is the same as the above result.

V.—ON THE TRANSFORMATION OF DEFINITE INTEGRALS.

By George Boole.

1. Jacobi, in the 15th vol. of *Crelle's Journal*, has proved the following remarkable transformation of a definite integral, quoted, with a demonstration, in Mr. Gregory's *Examples of the Diff. and Integ. Calculus*, p. 497, viz.

$$\int_0^\pi dx\, f^{(r)}(\cos x)(\sin x)^{2r} = 1.3.5..(2r-1)\int_0^\pi dx\, f(\cos x) \cos rx,$$

where $f^{(r)}(z) = \left(\dfrac{d}{dz}\right)^r f(z)$, and all the differential coefficients up to the $(r-1)^{\text{th}}$ inclusive, remain continuous from $x = 0$ to $x = \pi$. In the following pages I purpose to offer some development of a principle in analysis, which, while capable of being applied to various other questions, leads also to the theory of a class of transformations, of which the above is but a particular example.

2. By the well-known relation connecting the first and second of the Eulerian integrals,

$$\int_0^1 dz\, z^{l-1}(1-z)^{m-1} = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\int_0^1 dz\, z^{l'-1}(1-z)^{m'-1} = \frac{\Gamma(l')\,\Gamma(m')}{\Gamma(l'+m')};$$

hence

$$\int_0^1 dz\, z^{l-1}(1-z)^{m-1} = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(l'+m')}{\Gamma(l')\,\Gamma(m')\,\Gamma(l+m)}\int_0^1 dz\, z^{l'-1}(1-z)^{m'-1} \ldots (2),$$

l, m, l', m', being positive constant quantities. The principle on which our investigation will rest is simply this,—that the theorem (2) remains equally true, whether l, m, l', m', represent constant quantities, or symbols of operation, combining according to the same laws, and admitting, under the particular conditions of the question, of the same interpretation.

3. Let us now consider the definite integral,

$$\int_0^1 dz\, z^{l-1}(1-z)^{m-1} f(z).$$

This may be written under the form $\int_0^1 dz\, z^{l-1}(1-z)^{m-1} f(\varepsilon^{\theta} z)$, provided that $\varepsilon^{\theta} = 1$. Let $z = \varepsilon^{\phi}$, then

$$\int_0^1 dz\, z^{l-1}(1-z)^{m-1} f(\varepsilon^{\theta} z) = \int_0^1 dz\, z^{l-1}(1-z)^{m-1} f(\varepsilon^{\theta+\phi})$$

$$= \int_0^1 dz\, z^{l-1}(1-z)^{m-1} \varepsilon^{\phi\frac{d}{d\theta}} f(\varepsilon^{\theta}),$$

by Taylor's theorem. Replacing ε^{ϕ} by z, this gives

$$\int_0^1 dz\, z^{l-1}(1-z)^{m-1} f(\varepsilon^{\theta} z) = \int_0^1 dz\, z^{\frac{d}{d\theta}+l-1}(1-z)^{m-1} f(\varepsilon^{\theta}) \ldots\ldots(3).$$

Similarly we have

$$\int_0^1 dz\, z^{l'-1}(1-z)^{m'-1} f(\varepsilon^{\theta} z) = \int_0^1 dz\, z^{\frac{d}{d\theta}+l'-1}(1-z)^{m'-1} f(\varepsilon^{\theta}).$$

Now by theorem (2), and under the limitations implied in the principle above enunciated, the right-hand members of these two equations are connected by the relation

$$\int_0^1 dz\, z^{\frac{d}{d\theta}+l-1}(1-z)^{m-1} f(\varepsilon^{\theta})$$

$$= \frac{\Gamma\left(\frac{d}{d\theta}+l\right)\Gamma(m)\Gamma\left(\frac{d}{d\theta}+l'+m'\right)}{\Gamma\left(\frac{d}{d\theta}+l'\right)\Gamma(m')\Gamma\left(\frac{d}{d\theta}+l+m\right)} \int_0^1 dz\, z^{\frac{d}{d\theta}+l'-1}(1-z)^{m'-1} f(\varepsilon^{\theta}) \ldots(4);$$

hence, putting the first members respectively in the room of the second,

$$\int_0^1 dz\, z^{l-1}(1-z)^{m-1} f(\varepsilon^{\theta} z)$$

$$= \frac{\Gamma\left(\frac{d}{d\theta}+l\right)\Gamma(m)\Gamma\left(\frac{d}{d\theta}+l'+m'\right)}{\Gamma\left(\frac{d}{d\theta}+l'\right)\Gamma(m')\Gamma\left(\frac{d}{d\theta}+l+m\right)} \int_0^1 dz\, z^{l'-1}(1-z)^{m'-1} f(\varepsilon^{\theta} z)$$

$$= \int_0^1 dz\, z^{l'-1}(1-z)^{m'-1} \frac{\Gamma\left(\frac{d}{d\theta}+l\right)\Gamma(m)\Gamma\left(\frac{d}{d\theta}+l'+m'\right)}{\Gamma\left(\frac{d}{d\theta}+l'\right)\Gamma(m')\Gamma\left(\frac{d}{d\theta}+l+m\right)} f(\varepsilon^{\theta} z) \ldots(5).$$

4. Now to shew the conditions under which $\frac{d}{d\theta}$ admits of the required interpretation, we observe, that if $f(z)$ can be developed in positive ascending powers of z, we shall have, on effecting the development in the second member of the above equation, a series of terms of the form

$$A_i \int_0^1 dz\, z^{i+l'-1}(1-z)^{m'-1} \frac{\Gamma\left(\frac{d}{d\theta}+l\right)\Gamma(m)\Gamma\left(\frac{d}{d\theta}+l'+m'\right)}{\Gamma\left(\frac{d}{d\theta}+l'\right)\Gamma(m')\Gamma\left(\frac{d}{d\theta}+l+m\right)} \varepsilon^{i\theta} \ldots (6);$$

to each of which there will exist a corresponding term with the same constant coefficient, A_i, in the development of the first member. But by a known theorem,

$$\psi\left(\frac{d}{d\theta}\right)\varepsilon^{i\theta} = \psi(i)\,\varepsilon^{i\theta};$$

hence in each term the symbol $\frac{d}{d\theta}$ may be interpreted by a positive constant quantity, and the theorem verified by successive applications of (2). It is therefore true for all forms of $f(z)$ which can be expanded without negative indices.

In (5) again write $f(\varepsilon^{\theta+\phi})$ for $f(\varepsilon^{\theta}z)$, and observing that

$$\psi\left(\frac{d}{d\theta}\right)f(\varepsilon^{\theta+\phi}) = \psi\left(\frac{d}{d\phi}\right)f(\varepsilon^{\theta+\phi}) = \psi\left(\frac{d}{d\phi}\right)f(\varepsilon^{\phi}),$$

since $\varepsilon^{\theta}=1$, we have

$$\int_0^1 dz\, z^{l-1}(1-z)^{m-1} f(z)$$

$$= \int_0^1 dz\, z^{l'-1}(1-z)^{m'-1} \frac{\Gamma\left(\frac{d}{d\phi}+l\right)\Gamma(m)\Gamma\left(\frac{d}{d\phi}+l'+m'\right)}{\Gamma\left(\frac{d}{d\phi}+l'\right)\Gamma(m')\Gamma\left(\frac{d}{d\phi}+l+m\right)} f(\varepsilon^{\phi}) \ldots (7),$$

with the relation $\varepsilon^{\phi}=z$.

5. In applying the above theorem, let us first suppose that $l-l'$ and $m-m'$ are positive integers; then from the known relations,

$$\psi\left(\frac{d}{d\phi}+a\right)f(\varepsilon^{\phi}) = \varepsilon^{-a\phi}\,\psi\left(\frac{d}{d\phi}\right)\varepsilon^{a\phi} f(\varepsilon^{\phi}) \ldots\ldots (8),$$

$$\frac{d}{d\phi}\left(\frac{d}{d\phi}-1\right)\ldots\left(\frac{d}{d\phi}-m+1\right)f(\varepsilon^{\phi}) = z^{m}\left(\frac{d}{dz}\right)^{m} f(z) \ldots\ldots (9),$$

we easily find

$$\frac{\Gamma\left(\dfrac{d}{d\phi}+l\right)}{\Gamma\left(\dfrac{d}{d\phi}+l'\right)}f(\varepsilon^{\phi}) = \varepsilon^{-(l-1)\phi}\,\frac{\Gamma\left(\dfrac{d}{d\phi}+1\right)}{\Gamma\left\{\dfrac{d}{d\phi}-(l-l')+1\right\}}f(\varepsilon^{\phi})$$

$$= \varepsilon^{-(l-1)\phi}\,\frac{d}{d\phi}\left(\frac{d}{d\phi}-1\right)\cdot\cdot\left\{\frac{d}{d\phi}-(l-l')+1\right\}f(\varepsilon^{\phi})$$

$$= z^{-(l-1)}\,z^{l-l'}\left(\frac{d}{dz}\right)^{l-l'} z^{l-1}f(z)$$

$$= z^{-(l'-1)}\left(\frac{d}{dz}\right)^{l-l'} z^{l-1}f(z)\ \ldots\ldots\ldots\ (10);$$

and, by similar reasoning,

$$\frac{\Gamma\left(\dfrac{d}{d\phi}+l'+m'\right)}{\Gamma\left(\dfrac{d}{d\phi}+l+m\right)}f(\varepsilon^{\phi}) = z^{-(l+m-1)}\left(\frac{d}{dz}\right)^{l'+m'-l-m} z^{l'+m'-1}f(z)\ldots(11).$$

Substitute these forms in (7), and there will result

$$\int_0^1 dz\,z^{l-1}(1-z)^{m-1}f(z)$$

$$= \frac{\Gamma(m)}{\Gamma(m')}\int_0^1 dz\,(1-z)^{m'-1}\left(\frac{d}{dz}\right)^{l-l'} z^{-m}\left(\frac{d}{dz}\right)^{l'+m'-l-m} z^{l'+m'-1}f(z)\ \ldots\ (12),$$

a transformation of great generality, from which many particular results of an interesting character may be obtained.

It is obvious that the above theorem is equally true, when one or both the indices of $\dfrac{d}{dz}$ are fractional; but in this case it would be necessary, in the interpretation of our symbols, to revert to (7).

6. When the index of $\dfrac{d}{dz}$ is a negative integer, it must be observed that the symbol $\left(\dfrac{d}{dz}\right)^{-1}$ applied to a function developable in ascending positive powers of z, is equivalent to integration between the limits 0 and z; but if it should happen, which in the above theorem it cannot, that the development involves only negative powers of z, then the limits are $\pm\,\infty$, and z; for $\left(\dfrac{d}{dz}\right)^{-1}$ is evidently to be interpreted as the inverse of $\dfrac{d}{dz}$: now $\dfrac{d}{dz}z^{m} = mz^{m-1}$, therefore $\left(\dfrac{d}{dz}\right)^{-1} mz^{m-1} = z^{m}$, which is only true on the above assumption relative to the limits.

7. As particular illustrations of (12), let $l = l' = m' = \frac{1}{2}$, m being any positive constant, then

$$\int_0^1 dz\, z^{-\frac{1}{2}}(1-z)^{m-1} f(z) = \frac{\Gamma(m)}{\Gamma(\frac{1}{2})}\int_0^1 dz\, z^{-m}(1-z)^{-\frac{1}{2}}\left(\frac{d}{dz}\right)^{-(m-\frac{1}{2})} f(z).$$

As the index of $\dfrac{d}{dz}$ must be an integer, let $m - \frac{1}{2} = r$, we have

$$\int_0^1 dz\, z^{-\frac{1}{2}}(1-z)^{r-\frac{1}{2}} f(z) = \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^1 dz\, z^{-r+\frac{1}{2}}(1-z)^{-\frac{1}{2}}\left(\frac{d}{dz}\right)^{-r} f(z).$$

Put $z = (\cos x)^2$, then

$$(1-z) = (\sin x)^2, \text{ and } -\tfrac{1}{2}dz\, z^{-\frac{1}{2}}(1-z)^{\frac{1}{2}} = dx;$$

the limits inverted are 0 and $\frac{1}{2}\pi$, whence

$$\int_0^{\frac{1}{2}\pi} dx\,(\sin x)^{2r} f\{(\cos x)^2\} = \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^{\frac{1}{2}\pi} dx\, \frac{\left(\frac{d}{dz}\right)^{-r} f(z)}{(\cos x)^{2r}} \ldots (13),$$

provided that the integrations implied by the symbol $\left(\dfrac{d}{dz}\right)^{-r}$, be taken between the limits 0 and $(\cos x)^2$, $f(z)$ being developable in ascending positive powers of z.

Another form of the theorem will be obtained by assuming $\dfrac{1}{z^r}\left(\dfrac{d}{dz}\right)^{-r} f(z) = F(z)$, then

$$\int_0^{\frac{1}{2}\pi} dx\,(\sin x)^{2r}\left(\frac{d}{dz}\right)^{r}\{z\,F(z)\} = \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^{\frac{1}{2}\pi} dx\, F\{(\cos x)^2\} \ldots (14),$$

provided that after the differentiations are performed we change z into $(\cos x)^2$, and that $F(z)$ have no negative indices in its development, and, with its differential coefficients up to the $(r-1)^{\text{th}}$, is continuous within the limits of integration.

8. Again, in (12) let $l' = m$, $m' = l$, we have

$$\int_0^1 dz\, z^{l-1}(1-z)^{m-1} f(z) = \frac{\Gamma(m)}{\Gamma(l)}\int_0^1 dz\,(1-z)^{l-1}\left(\frac{d}{dz}\right)^{l-m} z^{l-1} f(z),$$

put $z^{l-1} f(z) = F(z)$, then

$$\int_0^1 dz\,(1-z)^{m-1} F(z) = \frac{\Gamma(m)}{\Gamma(l)}\int_0^1 dz\,(1-z)^{l-1}\left(\frac{d}{dz}\right)^{l-m} F(z).$$

If we further make $m = 1$, and for $l - 1$ write l, we find, on transposing the members,

$$\int_0^1 dz\,(1-z)^{l}\left(\frac{d}{dz}\right)^{l} F(z) = \Gamma(l+1)\int_0^1 dz\, F(z).$$

From the relation between $f(z)$ and $F(z)$, it appears that the latter function cannot involve in its development terms whose indices are lower than l. If any such occur, they

must be considered separately. In this way we find for the complete form of the theorem,

$$\int_0^1 dz\,(1-z)^l \left(\frac{d}{dz}\right)^l F(z)$$

$$=\Gamma(l+1)\left\{-F(o)-\frac{F'(o)}{1.2}\ldots-\frac{F^{(l-1)}(o)}{1.2\ldots l}+\int_0^1 dz\,F(z)\right\}\ldots(15),$$

true whenever $F(z)$ is a function developable by Taylor's theorem.

9. When $l-l'$, and $m-m'$ are not integers, the theorem (7) must be transformed by the introduction of additional symbols of definite integration. The general result is inconveniently long, but the method to be pursued will be sufficiently illustrated by the following examples.

Let $l=l'=m'=\frac{1}{2}$, and m be any positive constant, we have

$$\int_0^1 dz\,z^{-\frac{1}{2}}(1-z)^{m-1}f(\epsilon^\phi)=\int_0^1 dz\,z^{-\frac{1}{2}}(1-z)^{-\frac{1}{2}}\frac{\Gamma\left(\frac{d}{d\phi}+1\right)}{\Gamma\left(\frac{d}{d\phi}+m+\frac{1}{2}\right)}f(\epsilon^\phi)\ldots(16),$$

$$\int_0^1 dz\,z^{-\frac{1}{2}}(1-z)^{m-1}\,\Gamma\left(\frac{d}{d\phi}+m+\tfrac{1}{2}\right)f(\epsilon^\phi)$$

$$=\int_0^1 dz\,z^{-\frac{1}{2}}(1-z)^{-\frac{1}{2}}\,\Gamma\left(\frac{d}{d\phi}+1\right)f(\epsilon^\phi)\ldots(17).$$

Now $\Gamma\left(\frac{d}{d\phi}+m+\frac{1}{2}\right)f(\epsilon^\phi)=\int_0^\infty dv\,\epsilon^{-v}\,v^{\left(\frac{d}{d\phi}+m-\frac{1}{2}\right)}f(\epsilon^\phi)$, but $v^{\frac{d}{d\phi}}f(\epsilon^\phi)=f(\epsilon^\phi v)=f(vz)$; therefore

$$\Gamma\left(\frac{d}{d\phi}+m+\tfrac{1}{2}\right)f(\epsilon^\phi)=\int_0^\infty dv\,\epsilon^{-v}\,v^{m-\frac{1}{2}}f(vz).$$

Similarly $\Gamma\left(\frac{d}{d\phi}+1\right)f(\epsilon^\phi)=\int_0^\infty dv\,\epsilon^{-v}f(vz)$;

whence, substituting in (16),

$$\int_0^\infty dv\int_0^1 dz\,\epsilon^{-v}\,v^{m-\frac{1}{2}}\,z^{-\frac{1}{2}}(1-z)^{m-1}f(vz)$$

$$=\frac{\Gamma(m)}{\Gamma(\frac{1}{2})}\int_0^\infty dv\int_0^1 dz\,\epsilon^{-v}\,z^{-\frac{1}{2}}(1-z)^{-\frac{1}{2}}f(vz).$$

Put, as before, $m-\frac{1}{2}=r$ and $z=(\cos x)^2$, then

$$\int_0^\infty dv\int_0^{\frac{1}{2}\pi} dx\,\epsilon^{-v}\,v^r(\sin x)^{2r}f(v\cos x^2)$$

$$=\frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^\infty dv\int_0^{\frac{1}{2}\pi} dx\,\epsilon^{-v}f\{v(\cos x)^2\}\ldots.(18),$$

which is true for all values of r from $-\frac{1}{2}$ to ∞.

Again, in (7) let $l' = m$, $m' = l = 1$, and proceeding as above, we finally obtain

$$\int_0^\infty dv \int_0^1 dz\, \varepsilon^{-v}(v - vz)^{m-1} f(vz) = \Gamma(m) \int_0^\infty dv \int_0^1 dz\, \varepsilon^{-v} z^{m-1} f(vz) \ldots (19).$$

Both the preceding cases admit of another and more convenient form of solution, which I shall exemplify in the first. Thus, in (16),

$$\frac{\Gamma\left(\dfrac{d}{d\phi} + 1\right)}{\Gamma\left(\dfrac{d}{d\phi} + m + \frac{1}{2}\right)} f(\varepsilon^\phi) = \frac{1}{\Gamma(m - \frac{1}{2})} \frac{\Gamma\left(\dfrac{d}{d\phi} + 1\right)\Gamma(m - \frac{1}{2})}{\Gamma\left(\dfrac{d}{d\phi} + m + \frac{1}{2}\right)} f(\varepsilon^\phi) \ldots (20)$$

$$= \frac{1}{\Gamma(m - \frac{1}{2})} \int_0^1 dv\, v^{\frac{d}{d\phi}} (1 - v)^{m - \frac{3}{2}} f(\varepsilon^\phi) \text{ by (1)},$$

$$= \frac{1}{\Gamma(m - \frac{1}{2})} \int_0^1 dv\, (1 - v)^{m - \frac{3}{2}} f(vz);$$

whence, on substitution,

$$\int_0^1 dz\, z^{-\frac{1}{2}} (1 - z)^{m-1} f(z)$$

$$= \frac{\Gamma(m)}{\Gamma(\frac{1}{2})\, \Gamma(m - \frac{1}{2})} \int_0^1 dv \int_0^1 dz\, (1 - v)^{m - \frac{3}{2}} z^{-\frac{1}{2}} (1 - z)^{-\frac{1}{2}} f(vz):$$

put $m - \frac{1}{2} = r$, $v = (\cos y)^2$, $z = (\cos x)^2$, we find

$$\int_0^{\frac{1}{2}\pi} dx\, (\sin x)^{2r} f\{(\cos x)^2\}$$

$$= -\frac{2\Gamma(r + \frac{1}{2})}{\Gamma(r)\, \Gamma(\frac{1}{2})} \int_0^{\frac{1}{2}\pi} dx \int_0^{\frac{1}{2}\pi} dy\, \cos y\, (\sin y)^{2r-1} f\{(\cos x \cos y)^2\}.$$

This may evidently be written under the form

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} dx\, dy\, \cot y\, (\sin y)^{2r} f(\cos x \cos y)$$

$$= -\frac{\Gamma(r)\, \Gamma(\frac{3}{2})}{\Gamma(r + \frac{1}{2})} \int_0^{\frac{1}{2}\pi} dx\, (\sin x)^{2r} f(\cos x)$$

$$= -\frac{\Gamma(r)\, \Gamma(\frac{3}{2})}{\Gamma(r + \frac{1}{2})} \int_0^{\frac{1}{2}\pi} dy\, (\sin y)^{2r} f(\cos y). \ldots (21).$$

In a similar way we shall find

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} dx\, dy\, (\cos x)^{2r} \tan x\, f(\sin x \sin y)$$

$$= \frac{\Gamma(r)\, \Gamma(\frac{3}{2})}{\Gamma(r + \frac{1}{2})} \int_0^{\frac{1}{2}\pi} dx\, (\cos x)^{2r} f(\sin x). \ldots (22).$$

In the last theorem let $r = \frac{1}{2}$, and in the second member let $\sin x = v$, then the transformed limits being 0 and 1,

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} dx\, dy \sin x\, f(\sin x \sin y) = \tfrac{1}{2}\pi \int_0^1 dv\, f(v) \ldots . (23),$$

a very remarkable theorem, enabling us at once to assign the values of a large number of definite double integrals. It must be observed that, as before, $f(v)$ must not involve in its development negative powers of v.

10. Many particular results of great interest may be obtained without the aid of the general theorem (7). Thus from the first Eulerian integral, $\int_0^\infty \varepsilon^{-x} x^{n-1} dx = \Gamma(n)$, treated by the method of section 3, we find

$$\int_0^\infty dx\, \varepsilon^{-x} \left(\frac{d}{dx}\right)^{-r} F(x) = \int_0^\infty dx\, \varepsilon^{-x} F(x),$$

a theorem true for all positive values of r. If r be an integer, and $F(x)$ a function developable by Taylor's theorem, we find, by reasoning similar to that of section 8,

$$\int_0^\infty dx\, \varepsilon^{-x} F^{(r)} x = -\{F(o) + F^{(1)}(o) \ldots + F^{(r-1)}(o)\} + \int_0^\infty dx\, \varepsilon^{-x} F(x) \ldots (24).$$

From the definite integral $\int_0^\pi dx\, \varepsilon^{ax} (\cos x)^n$, we obtain the remarkable theorems

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} dx\, (\cos x)^{2r} F(x) = \Gamma(2r+1) \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} dx\, \phi(x) \ldots . (25),$$

$$\int_0^\pi dx\, (\sin x)^{2r} F(x) = \Gamma(2r+1) \int_0^\pi dx\, \phi(x) \ldots . (26),$$

where $F(x) = \left(\frac{d^2}{dx^2} + 2^2\right)\left(\frac{d^2}{dx^2} + 4^2\right) \ldots \left\{\frac{d^2}{dx^2} + (2r)^2\right\} \phi x;$

theorems which are generally true, whether the development of $\phi(x)$ be free from negative indices or not.

To enter upon the general illustration of the above theorems, would extend this paper beyond its proper limits: two or three examples must therefore suffice. For this purpose I select the last two theorems, (25) and (26).

In (26), let $\phi x = \cos mx$; then, by a known theorem, $\psi\left(\frac{d^2}{dx^2}\right) \cos mx = \psi(-m^2) \cos mx$, whence

$$\int_0^\pi (\sin x)^{2r} \cos mx = \frac{\Gamma(2r+1)}{(2^2 - m^2)(4^2 - m^2) \ldots \{(2r)^2 - m^2\}} \frac{\sin m\pi}{m}.$$

In (26) again, let $r = 2$, then

$$\int_0^\pi (\sin x)^2 \left(\frac{d^2}{dx^2} + 4\right) \phi x\, dx = 2\int_0^\pi \phi(x)\, dx \ldots (27).$$

Let $\phi(x) = \dfrac{1}{1+x}$, whence $\displaystyle\int_0^\pi dx\, \phi(x) = \log(1+\pi)$, and effecting the differentiations in the first member, we find

$$\int_0^\pi dx\, (\sin x)^2\, \frac{2 + 2x + x^2}{(1+x)^3} = \log(1+\pi);$$

and similarly, from (25),

$$\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} dx\, (\cos x)^2\, \frac{2 + 2x + x^2}{(1+x)^3} = \log\left(\frac{\pi^2}{4} - 1\right).$$

Let $\phi(x) = x^m$, then by (27), m being $> (-1)$

$$\int_0^\pi dx \sin x^2 \left\{x^m + \frac{m(m-1)\, x^{m-2}}{4}\right\} = \frac{\pi^{m+1}}{m+1}.$$

The above results I have verified independently.

VI.—ON THE MOTION OF ROTATION OF A SOLID BODY.

By Arthur Cayley, B.A. Fellow of Trinity College.

In the fifth volume of Liouville's Journal, in a paper "Des lois géometriques qui regissent les deplacemens d'un systeme solide," M. Olinde Rodrigues has given some very elegant formulæ for determining the position of two sets of rectangular axes with respect to each other, employing rational functions of three quantities only. The principal object of the present paper is to apply these to the problem of the rotation of a solid body; but I shall first demonstrate the formulæ in question, and some others connected with the same subject which may be useful on other occasions.

Let Ax, Ay, Az; $Ax_{\prime}$, $Ay_{\prime}$, $Az_{\prime}$, be any two sets of rectangular axes passing through the point A, x, y, z; $x_{\prime}$, $y_{\prime}$, $z_{\prime}$, being taken for the points where these lines intersect the spherical surface described round the centre A with radius unity. Join $xx_{\prime}$, $yy_{\prime}$, $zz_{\prime}$, by arcs of great circles, and through the central points of these describe great circles cutting them at right angles. These are easily seen to intersect in a certain point P. Let $Px = f$, $Py = g$, $Pz = h$; then also $Px_{\prime} = f$, $Py_{\prime} = g$, $Pz_{\prime} = h$. And $\angle xPx_{\prime} = \angle yPy_{\prime} = \angle zPz_{\prime} = \theta$ suppose, θ being measured from xP towards yP, yP towards zP, or

zP towards xP. The cosines of f, g, h, are of course connected by the equation

$$\cos^2 f + \cos^2 g + \cos^2 h = 1.$$

Let α, β, γ; α', β', γ'; $\alpha'', \beta'', \gamma''$, represent the cosines of $x_{\prime}x, y_{\prime}x, z_{\prime}x$; $x_{\prime}y, y_{\prime}y, z_{\prime}y$; $x_{\prime}z, y_{\prime}z, z_{\prime}z$: these quantities are to be determined as functions of f, g, h, θ.

Suppose for a moment,

$$\angle yPz = \mathrm{x}, \quad \angle zPx = \mathrm{y}, \quad \angle xPy = \mathrm{z}.$$

Then

$$\begin{aligned}
\alpha &= \cos^2 f + \sin^2 f \cos\theta,\\
\alpha' &= \cos f \cos g + \sin f \sin g \cos\{(\mathrm{z} - \theta)\},\\
\alpha'' &= \cos f \cos h + \sin f \sin h \cos\{(\mathrm{y} + \theta)\}.\\
\beta &= \cos g \cos f + \sin g \sin f \cos\{(\mathrm{z} + \theta)\},\\
\beta' &= \cos^2 g + \sin^2 g \cos\theta,\\
\beta'' &= \cos g \cos h + \sin g \sin h \cos\{(\mathrm{x} - \theta)\}.\\
\gamma &= \cos h \cos f + \sin h \sin f \cos\{(\mathrm{y} - \theta)\},\\
\gamma' &= \cos h \cos g + \sin h \sin g \cos\{(\mathrm{x} + \theta)\},\\
\gamma'' &= \cos^2 h + \sin^2 h \cos\theta.
\end{aligned}$$

Also

$$\begin{aligned}
\sin g \sin h \cos \mathrm{x} &= -\cos g \cos h,\\
\sin h \sin f \cos \mathrm{y} &= -\cos h \cos f,\\
\sin f \sin g \cos \mathrm{z} &= -\cos f \cos g;
\end{aligned}$$

and

$$\begin{aligned}
\sin g \sin h \sin \mathrm{x} &= \cos f,\\
\sin h \sin f \sin \mathrm{y} &= \cos g,\\
\sin f \sin g \sin \mathrm{z} &= \cos h.
\end{aligned}$$

Substituting,

$$\begin{aligned}
\alpha &= \cos^2 f + \sin^2 f \cos\theta,\\
\alpha' &= \cos f \cos g\,(1 - \cos\theta) + \cos h \sin\theta,\\
\alpha'' &= \cos f \cos h\,(1 - \cos\theta) - \cos g \sin\theta.\\
\beta &= \cos g \cos f\,(1 - \cos\theta) - \cos h \sin\theta,\\
\beta' &= \cos^2 g + \sin^2 g \cos\theta,\\
\beta'' &= \cos g \cos h\,(1 - \cos\theta) + \cos f \sin\theta.\\
\gamma &= \cos h \cos f\,(1 - \cos\theta) + \cos g \sin\theta,\\
\gamma' &= \cos h \cos g\,(1 - \cos\theta) - \cos f \sin\theta,\\
\gamma'' &= \cos^2 h + \sin^2 h \cos\theta.
\end{aligned}$$

Assume $\lambda = \tan\frac{1}{2}\theta \cos f$, $\mu = \tan\frac{1}{2}\theta \cos g$, $\nu = \tan\frac{1}{2}\theta \cos h$, and $\sec^2\frac{1}{2}\theta = 1 + \lambda^2 + \mu^2 + \nu^2 = \kappa$. Then

$$\begin{array}{lll}
\kappa\alpha = 1 + \lambda^2 - \mu^2 - \nu^2, & \kappa\alpha' = 2(\lambda\mu + \nu), & \kappa\alpha'' = 2(\nu\lambda - \mu),\\
\kappa\beta = 2(\lambda\mu - \nu), & \kappa\beta' = 1 + \mu^2 - \nu^2 - \lambda^2, & \kappa\beta'' = 2(\mu\nu + \lambda);\\
\kappa\gamma = 2(\nu\lambda + \mu), & \kappa\gamma' = 2(\mu\nu - \lambda), & \kappa\gamma'' = 1 + \nu^2 - \lambda^2 - \mu^2;
\end{array}$$

which are the formulæ required, differing only from those in Liouville, by having λ, μ, ν, instead of $\frac{1}{2}m, \frac{1}{2}n, \frac{1}{2}p$; and a, a', a''; β, β', β''; $\gamma, \gamma', \gamma''$, instead of a, b, c; a', b', c'; a'', b'', c''. It is to be remarked, that β', β'', β; $\gamma'', \gamma, \gamma'$, are deduced from a, a', a'', by writing μ, ν, λ; ν, λ, μ, for λ, μ, ν.

Let $1 + a + \beta' + \gamma'' = v$;

$$\kappa v = 4,$$

$$\lambda v = \beta'' - \gamma', \qquad \mu v = \gamma - a'', \qquad \kappa v = a' - \beta,$$

$$\lambda^2 v = 1 + a - \beta' - \gamma'', \quad \mu^2 v = 1 - a + \beta' - \gamma'', \quad \nu^2 v = 1 - a - \beta' - \gamma''.$$

Suppose that Ax, Ay, Az, are referred to axes $A\text{X}, A\text{Y}, A\text{Z}$, by the quantities l, m, n, k, analogous to $\lambda, \mu, \nu, \kappa$, these latter axes being referred to $Ax_{,}, Ay_{,}, Az_{,}$, by the quantities $l_{,}, m_{,}, n_{,}, k_{,}$.

Let a, b, c; a', b', c'; a'', b'', c''; $a_{,}, b_{,}, c_{,}$; $a_{,}', b_{,}', c_{,}'$; $a_{,}'', b_{,}'', c_{,}''$, denote the quantities analogous to a, β, γ; a', β', γ'; a'', β'', γ''. Then we have, by spherical trigonometry, the formulæ

$$a = aa_{,} + ba_{,}' + ca_{,}'', \qquad \beta = ab_{,} + bb_{,}' + cb_{,}'', \qquad \gamma = ac_{,} + bc_{,}' + cc_{,}'';$$

$$a' = a'a_{,} + b'a_{,}' + c'a_{,}'', \qquad \beta' = a'b_{,} + b'b_{,}' + c'b_{,}'', \qquad \gamma' = a'c_{,} + b'c_{,}' + c'c_{,}'';$$

$$a'' = a''a_{,} + b''a_{,}' + c''a_{,}'', \qquad \beta'' = a''b_{,} + b''b_{,}' + c''b_{,}'', \qquad \gamma'' = a''c_{,} + b''c_{,}' + c''c_{,}''.$$

Then expressing a, b, c; a', b', c'; a'', b'', c''; $a_{,}, b_{,}, c_{,}$; $a_{,}', b_{,}', c_{,}'$; $a_{,}'', b_{,}'', c_{,}''$, in terms of l, m, n; $l_{,}, m_{,}, n_{,}$, after some reductions we arrive at

$$kk_{,}v = 4(1 - ll_{,} - mm_{,} - nn_{,})^2 = 4\Pi^2 \text{ suppose},$$

$$kk_{,}(\beta'' - \gamma) = 4(l + l_{,} + n_{,}m - nm_{,})\,\Pi,$$

$$kk_{,}(\gamma - a') = 4(m + m_{,} + l_{,}m - lm_{,})\,\Pi,$$

$$kk_{,}(a' - \beta'') = 4(n + n_{,} + m_{,}n - mn_{,})\,\Pi.$$

And hence

$$\Pi = 1 - ll_{,} - mm_{,} - nn_{,}, \qquad \Pi\lambda = l + l_{,} + n_{,}m - nm_{,},$$

$$\Pi\mu = m + m_{,} + l_{,}m - lm_{,}, \qquad \Pi\nu = n + n_{,} + m_{,}n - mn_{,},$$

which are formulæ of considerable elegance for exhibiting the combined effect of successive displacements of the axes. The following analogous ones are readily obtained:

$$P = 1 + \lambda l + \mu m + \nu n, \qquad Pl_{,} = \lambda - l - \nu m + \mu n,$$

$$Pm_{,} = \mu - m - \lambda n + \nu l, \qquad Pn_{,} = \nu - n - \mu l + \lambda m:$$

and again,

$$P_{,} = 1 + \lambda l_{,} + \mu m_{,} + \nu n_{,}, \qquad P_{,}l = \lambda - l_{,} + \nu m_{,} - \mu n_{,},$$

$$P_{,}m = \mu - m_{,} + \lambda n_{,} - \nu l_{,}, \qquad P_{,}n = \nu - n_{,} + \mu l_{,} - \lambda m_{,}.$$

These formulæ will be found useful in the integration of the equations of rotation of a solid body.

Next it is required to express the quantities p, q, r, in terms of λ, μ, ν, where as usual

$$p = \gamma \frac{d\beta}{dt} + \gamma' \frac{d\beta'}{dt} + \gamma'' \frac{d\beta''}{dt},$$

$$q = \alpha \frac{d\gamma}{dt} + \alpha' \frac{d\gamma'}{dt} + \alpha'' \frac{d\gamma''}{dt},$$

$$r = \beta \frac{d\alpha}{dt} + \beta' \frac{d\alpha'}{dt} + \beta'' \frac{d\alpha''}{dt}.$$

Differentiating the values of $\beta\kappa$, $\beta'\kappa$, $\beta''\kappa$, multiplying by $\gamma, \gamma', \gamma''$, and adding,

$$\kappa p = 2\lambda'(\gamma\mu - \gamma'\lambda + \gamma'') + 2\mu'(\gamma\lambda - \gamma'\mu + \gamma''\nu) + 2\nu'(-\gamma - \gamma'\nu + \gamma''\mu),$$

where λ', μ', ν', denote $\frac{d\lambda}{dt}, \frac{d\mu}{dt}, \frac{d\nu}{dt}$. Reducing, we have

$$\kappa p = 2(\lambda' + \nu\mu' - \nu'\mu):$$

from which it is easy to derive the system

$$\begin{aligned} \kappa p &= 2(\lambda' + \nu\mu' - \nu'\mu), \\ \kappa q &= 2(-\nu\lambda' + \mu' + \nu'\lambda), \\ \kappa r &= 2(\mu\lambda' - \mu'\lambda + \nu'); \end{aligned}$$

or, determining λ', μ', ν', from these equations, the equivalent system

$$\begin{aligned} 2\lambda' &= (1 + \lambda^2)p + (\lambda\mu - \nu)q + (\nu\lambda + \mu)r, \\ 2\mu' &= (\lambda\mu + \nu)p + (1 + \mu^2)q + (\mu\nu - \lambda)r, \\ 2\nu' &= (\nu\lambda - \mu)p + (\mu\nu + \lambda)q + (1 + \nu^2)r. \end{aligned}$$

The following equation also is immediately obtained,

$$\kappa' = \kappa(\lambda p + \mu q + \nu r).$$

The subsequent part of the problem requires the knowledge of the differential coefficients of p, q, r, with respect to λ, μ, ν; λ', μ', ν'. It will be sufficient to write down the six.

$$\kappa \frac{dp}{d\lambda'} = 2, \qquad \kappa \frac{dp}{d\lambda} + 2p\lambda = 0,$$

$$\kappa \frac{dq}{d\lambda'} = -2\nu, \qquad \kappa \frac{dq}{d\lambda} + 2q\lambda = 2\nu',$$

$$\kappa \frac{dr}{d\lambda'} = 2\mu, \qquad \kappa \frac{dr}{d\lambda} + 2r\lambda = -2\mu',$$

from which the others are immediately obtained.

Suppose now a solid body acted on by any forces, and revolving round a fixed point. The equations of motion are

$$\frac{d}{dt}\cdot\frac{dT}{d\lambda'}-\frac{dT}{d\lambda}=\frac{dV}{d\lambda},$$
$$\frac{d}{dt}\cdot\frac{dT}{d\mu'}-\frac{dT}{d\mu}=\frac{dV}{d\mu},$$
$$\frac{d}{dt}\cdot\frac{dT}{d\nu'}-\frac{dT}{d\nu}=\frac{dV}{d\nu}.$$

Where $T=\frac{1}{2}(Ap^2+Bq^2+Cr^2)$,

$$V=\Sigma\,[\int(Xdx+Ydy+Zdz)]\,dm.$$

Or if $Xdx+Ydy+Zdz$ is not an exact differential, $\frac{dV}{d\lambda}, \frac{dV}{d\mu}, \frac{dV}{d\nu}$, are independent symbols standing for

$$\Sigma\left(X\frac{dx}{d\lambda}+Y\frac{dy}{d\lambda}+Z\frac{dz}{d\lambda}\right)dm,\ldots.$$

Vide *Mécanique Analytique*, Avertissement, t. I. p. 4. Only in this latter case V stands for the disturbing function, the principal forces vanishing.

Now, considering the first of the above equations

$$\frac{dT}{d\lambda'}=Ap\frac{dp}{d\lambda}+Bq\frac{dq}{d\lambda}+Cr\frac{dr}{d\lambda}$$
$$=\frac{2}{\kappa}(Ap-\nu Bq+\mu Cr).$$

Whence, writing p', q', r', κ', for $\frac{dp}{dt}, \frac{dq}{dt}, \frac{dr}{dt}, \frac{d\kappa}{dt}$,

$$\frac{d}{dt}\left(\frac{dT}{d\lambda'}\right)=\frac{2}{\kappa}(Ap'-\nu Bq'+\mu Cr')-\frac{2}{\kappa}Bq\nu'$$
$$+\frac{2}{\kappa}Cr\mu'-\frac{2\kappa'}{\kappa^2}(Ap-\nu Bq+\mu Cr).$$

Also $\frac{dT}{d\lambda}=Ap\frac{dp}{d\lambda}+Bq\frac{dq}{d\lambda}+Cr\frac{dr}{d\lambda}$

$$=-\frac{2\lambda}{\kappa}(Ap^2+Bq^2+Cr^2)+\frac{2}{\kappa}Bq\nu'-\frac{2}{\kappa}Cr\mu'.$$

And hence

$$\frac{1}{2}\left(\frac{d}{dt}\ \frac{dT}{d\lambda'}-\frac{dT}{d\lambda}\right)=\frac{1}{\kappa}(Ap'-\nu Bq'+\mu Cr')-\frac{2}{\kappa}Bq\nu'+\frac{2}{\kappa}Cr\mu'$$
$$+\frac{\lambda}{\kappa}(Ap^2+Bq^2+Cr^2)-\frac{\kappa'}{\kappa^2}(Ap-\nu Bq+\mu Cr).$$

Substituting for $\lambda', \mu', \nu', \kappa'$, after all reductions,

$$\frac{1}{2}\left(\frac{d}{dt}\cdot\frac{dT}{d\lambda'}-\frac{dT}{d\lambda}\right)=\frac{1}{\kappa}[\{Ap'+(C-B)qr\}-\nu\,\{Bq'+(A-C)rp\}$$
$$+\mu\,\{Cr'+(B-A)pq\}].$$

And, forming the analogous quantities in μ, ν, and substituting in the equations of motion, these become

$$Ap'+(C-B)qr-\nu\{Bq'+(A-C)rp\}+\mu\{Cr'+(B-A)pq\}=\tfrac{1}{2}\kappa\frac{dV}{d\lambda},$$

$$\nu\{Ap'+(C-B)qr\}+\{Bq'+(A-C)rp\}-\lambda\{Cr'+(B-A)pq\}=\tfrac{1}{2}\kappa\frac{dV}{d\mu},$$

$$\mu\{Ap'+(C-B)qr\}+\lambda\{Bq'+(A-C)rp\}+\{Cr'+(B-A)pq\}=\tfrac{1}{2}\kappa\frac{dV}{d\nu}.$$

Or eliminating, and replacing p', q', r', by $\frac{dp}{dt}$, $\frac{dq}{dt}$, $\frac{dr}{dt}$,

$$A\frac{dp}{dt}+(C-B)qr=\frac{1}{2}\left\{(1+\lambda^2)\frac{dV}{d\lambda}+(\lambda\mu+\nu)\frac{dV}{d\mu}+(\nu\lambda-\mu)\frac{dV}{d\nu}\right\},$$

$$B\frac{dq}{dt}+(A-C)rp=\frac{1}{2}\left\{(\lambda\mu-\nu)\frac{dV}{d\lambda}+(1+\mu^2)\frac{dV}{d\mu}+(\mu\nu+\lambda)\frac{dV}{d\nu}\right\},$$

$$C\frac{dr}{dt}+(B-A)pq=\frac{1}{2}\left\{(\nu\lambda+\mu)\frac{dV}{d\lambda}+(\mu\nu-\lambda)\frac{dV}{d\mu}+(1+\nu^2)\frac{dV}{d\nu}\right\};$$

to which are to be joined

$$\kappa p=2\left(\frac{d\lambda}{dt}+\nu\frac{d\mu}{dt}-\mu\frac{d\nu}{dt}\right),$$

$$\kappa q=2\left(-\nu\frac{d\lambda}{dt}+\frac{d\mu}{dt}+\lambda\frac{d\nu}{dt}\right),$$

$$\kappa r=2\left(\mu\frac{d\lambda}{dt}-\lambda\frac{d\mu}{dt}+\frac{d\nu}{dt}\right);$$

where it will be recollected

$$\kappa=1+\lambda^2+\mu^2+\nu^2;$$

and on the integration of these six equations depends the complete determination of the motion.

If we neglect the terms depending on V, the first three equations may be integrated in the form

$$p^2=p_1^2-\frac{C-B}{A}\phi.\quad q^2=q_1^2-\frac{A-C}{B}\phi.\quad r^2=r_1^2-\frac{B-A}{C}\phi.$$

$$2t=\int\frac{d\phi}{\left\{\left(p_1^2-\frac{C-B}{A}\phi\right)\left(q_1^2-\frac{A-C}{B}\phi\right)\left(r_1^2-\frac{B-A}{C}\phi\right)\right\}^{\frac{1}{2}}}.$$

And considering p, q, r as functions of ϕ, given by these equations, the three latter ones take the form

$$\frac{\kappa}{4qr} = \frac{d\lambda}{d\phi} + \nu \frac{d\mu}{d\phi} - \mu \frac{d\nu}{d\phi},$$

$$\frac{\kappa}{4rp} = -\nu \frac{d\lambda}{d\phi} + \frac{d\mu}{d\phi} + \lambda \frac{d\nu}{d\phi},$$

$$\frac{\kappa}{4pq} = \mu \frac{d\lambda}{d\phi} - \lambda \frac{d\mu}{d\phi} + \frac{d\nu}{d\phi};$$

of which, as is well known, the equations following, equivalent to two independent equations, are integrals.

$$\kappa g = Ap\,(1 + \lambda^2 - \mu^2 - \nu^2) + 2Bq\,(\lambda\mu - \nu) + 2Cr\,(\nu\lambda + \mu),$$

$$\kappa g' = 2Ap\,(\lambda\mu + \nu) + Bq\,(1 + \mu^2 - \lambda^2 - \nu^2) + 2Cr\,(\mu\nu - \lambda),$$

$$\kappa g'' = 2Ap\,(\nu\lambda - \mu) + 2Bq\,(\mu\nu + \lambda) + Cr\,(1 + \nu^2 - \lambda^2 - \mu^2);$$

where g, g', g'', are arbitrary constants satisfying

$$g^2 + g'^2 + g''^2 = A^2p_1^{\ 2} + B^2q_1^{\ 2} + C^2r_1^{\ 2}.$$

To obtain another integral, it is apparently necessary, as in the ordinary theory, to revert to the consideration of the invariable plane. Suppose $g' = 0$, $g'' = 0$.

Then $\qquad g'' = \sqrt{(A^2p_1^{\ 2} + B^2q_1^{\ 2} + C^2r_1^{\ 2})} = k$ suppose.

We easily obtain, where $\lambda_0, \mu_0, \nu_0, \kappa_0$ are written for $\lambda, \mu, \nu, \kappa$, to denote this particular supposition,

$$\kappa_0 Ap = 2\,(\nu_0\lambda_0 - \mu_0)\,k,$$

$$\kappa_0 Bq = 2\,(\mu_0\nu_0 + \lambda_0)\,k,$$

$$\kappa_0 Cr = (1 + \nu_0^{\ 2} - \lambda_0^{\ 2} - \mu_0^{\ 2})\,k;$$

whence, and from $\quad \kappa_0 = 1 + \lambda_0^{\ 2} + \mu_0^{\ 2} + \nu_0^{\ 2}$,

$$\kappa_0 Cr = (2 + 2\nu_0^{\ 2} - \kappa_0)\,k,$$

$$\kappa_0 = \frac{(2 + 2\nu_0^{\ 2})\,k}{k + Cr}.$$

And therefore

$$\nu_0\lambda_0 - \mu_0 = \frac{(1 + \nu_0^{\ 2}) \,.\, Ap}{k + Cr}, \quad \mu_0\nu_0 + \lambda_0 = \frac{(1 + \nu_0^{\ 2}) \,.\, Bq}{k + Cr}.$$

Hence, writing $h = Ap_1^{\ 2} + Bq_1^{\ 2} + Cr_1^{\ 2}$, the equation

$$\frac{d\nu_0}{d\phi} = \frac{1}{4pqr}\left\{(\nu_0\lambda_0 - \mu_0)\,p + (\mu_0\nu_0 + \lambda_0)\,q + (1 + \nu^2)\,r\right\},$$

reduces itself to

$$\frac{4}{1 + \nu_0^{\ 2}} \cdot \frac{d\nu_0}{d\phi} = \frac{h + kr}{(k + Cr)\,pqr},$$

$$\text{or} \qquad 4 \tan^{-1} \nu_0 = \int \frac{(h + kr) \,.\, d\phi}{(k + Cr)\,pqr}.$$

The integral taking rather a simpler form if p, q, ϕ be considered functions of r, and becoming

$$2\tan^{-1}\nu_0 = \int \frac{h+kr}{k+Cr}\,\frac{C\sqrt{(AB)}\,.\,dr}{\sqrt{[\{k^2-Bh+(B-C)Cr^2\}\{Ah-k^2+(C-A)Cr^2\}]}};$$

and (ν_0) being determined, λ_0, μ_0, are given by the equations

$$\lambda_0 = \frac{\nu_0 Ap + Bq}{k + Cr}, \qquad \mu_0 = \frac{\nu_0 Bq - Ap}{k + Cr}.$$

Hence l, m, n, denoting arbitrary constants, the general values of λ, μ, ν, are given by the equations

$$\begin{aligned} P_0 &= 1 - l\lambda_0 - m\mu_0 - n\nu_0, \\ P_0\lambda &= l + \lambda_0 + \nu_0 m - \mu_0 n, \\ P_0\mu &= m + \mu_0 + \lambda_0 n - \nu_0 l, \\ P_0\nu &= n + \nu_0 + \mu_0 l - \lambda_0 m. \end{aligned}$$

In a following paper I propose to develope the formulæ for the variations of the arbitrary constants p_1, q_1, r_1, l, m, n, when the terms involving V are taken into account.

Note.—It may be as well to verify independently the analytical conclusion immediately deducible from the preceding formulæ, viz. if λ, μ, ν, be given by the differential equations,

$$\kappa p = \frac{d\lambda}{dt} + \nu\frac{d\mu}{dt} - \mu\frac{d\nu}{dt},$$

$$\kappa q = -\nu\frac{d\lambda}{dt} + \frac{d\mu}{dt} + \lambda\frac{d\nu}{dt},$$

$$\kappa r = \mu\frac{d\lambda}{dt} - \lambda\frac{d\mu}{dt} + \frac{d\nu}{dt},$$

where $\kappa = 1 + \lambda^2 + \mu^2 + \nu^2$, and p, q, r, are any functions of t. Then if λ_0, μ_0, ν_0, be particular values of λ, μ, ν, and l, m, n, arbitrary constants, the general integrals are given by the system

$$\begin{aligned} P_0 &= 1 - l\lambda_0 - m\mu_0 - n\nu_0, \\ P_0\lambda &= l + \lambda_0 + \nu_0 m - \mu_0 n, \\ P_0\mu &= m + \mu_0 + \lambda_0 n - \nu_0 l, \\ P_0\nu &= n + \nu_0 + \mu_0 l - \lambda_0 m. \end{aligned}$$

Assuming these equations, we deduce the equivalent system,

$$\begin{aligned} (1 + \lambda\lambda_0 + \mu\mu_0 + \nu\nu_0)\,l &= \lambda - \lambda_0 + \nu_0\mu - \nu\mu_0, \\ (1 + \lambda\lambda_0 + \mu\mu_0 + \nu\nu_0)\,m &= \mu - \mu_0 + \lambda_0\nu - \lambda\nu_0, \\ (1 + \lambda\lambda_0 + \mu\mu_0 + \nu\nu_0)\,n &= \nu - \nu_0 + \mu_0\lambda - \mu\lambda_0. \end{aligned}$$

Differentiate the first of these and eliminate l, the result takes the form

$$\begin{aligned} &-(\mu_0^2 + \nu_0^2)(\lambda' + \nu\mu' - \nu'\mu) - (\nu_0 - \lambda_0\mu_0)(-\nu\lambda' + \mu' + \lambda\nu') \\ &\qquad + (\mu_0 + \lambda_0\nu_0)(\mu\lambda' - \lambda\mu' + \nu') + \kappa_0\lambda', \\ &+(\mu^2 + \nu^2)(\lambda_0' + \nu_0\mu_0' - \nu_0'\mu_0) + (\nu - \lambda\mu)(-\nu_0\lambda_0' + \mu_0' + \lambda_0\nu_0') \\ &\qquad - (\mu + \lambda\nu)(\mu_0\lambda_0' - \lambda_0\mu_0' + \nu_0') - \kappa\lambda_0' = 0, \end{aligned}$$

Where λ', &c. denote $\dfrac{d\lambda}{dt}$, &c.

$$\kappa_0 = 1 + \lambda_0^2 + \mu_0^2 + \nu_0^2.$$

Reducing by the differential equations in λ, μ, ν; λ_0, μ_0, ν_0, this becomes

$$\kappa_0 \{\lambda' + \tfrac{1}{2}p(\mu^2 + \nu^2) + \tfrac{1}{2}q(\nu - \lambda\mu) - \tfrac{1}{2}r(\mu + \lambda\nu)\}$$
$$- \kappa\{\lambda_0' + \tfrac{1}{2}p(\mu_0^2 + \nu_0^2) + \tfrac{1}{2}q(\nu_0 - \lambda_0\mu_0) - \tfrac{1}{2}r(\mu_0 + \lambda\nu_0)\} = 0;$$

or substituting for λ', λ_0', this reduces itself to the identical equation

$$\tfrac{1}{2}p(\kappa_0\kappa - \kappa\kappa_0) = 0:$$

and similarly may the remaining equations be verified.

VII.—SOLUTION OF A PROBLEM IN ANALYTICAL GEOMETRY.

The following problem is given in *Grunert's Archiv der Mathematik und Physik*, I. 136, and as it has not yet found a place in any treatise on Analytical Geometry, it may be new to most of our readers. The demonstration differs from that of Grunert in the use of the symmetrical equations to the straight line; in other respects the method is the same.

Let the equations to the four given lines be

$$\frac{x-a}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \ldots (1), \quad \frac{x-a_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} \ldots (2),$$

$$\frac{x-a_2}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2} \ldots (3), \quad \frac{x-a_3}{l_3} = \frac{y-\beta_3}{m_3} = \frac{z-\gamma_3}{n_3} \ldots (4),$$

and assume the required equations to be

$$\frac{x-x'}{\lambda} = \frac{y-y'}{\mu} = \frac{z-z'}{\nu} \ldots\ldots (5).$$

Since x', y', z', are only limited to be the co-ordinates of some point in the line (5), we may assume them to be those of the intersection of (1) and (5), which gives us the equations

$$\frac{x'-a}{l} = \frac{y'-\beta}{m} = \frac{z'-\gamma}{n} = r \quad \text{suppose} \ldots\ldots (6).$$

But the conditions for the intersection of (5) and (2), of (5) and (3), and of (4) and (3) are

$$\left.\begin{aligned} L_1(x'-a_1) + M_1(y'-\beta_1) + N_1(z'-\gamma_1) = 0 \\ L_2(x'-a_2) + M_2(y'-\beta_2) + N_2(z'-\gamma_2) = 0 \\ L_3(x'-a_3) + M_3(y'-\beta_3) + N_3(z'-\gamma_3) = 0 \end{aligned}\right\} \ldots (7),$$

if we put for shortness

$$L_1 = m_1\nu - n_1\mu, \quad M_1 = n_1\lambda - l_1\nu, \quad N_1 = l_1\mu - m_1\lambda,$$

and similarly for the other quantities.

But $x' - a' = x' - a - (a_1 - a)$ and similarly for the others; hence, substituting in (7) the value of $x' - a$, $y' - \beta$, $z' - \gamma$, derived from (6) we have three equations of which the first is

$$L_1 \{lr - (a_1 - a)\} + M_1 \{mr - (\beta_1 - \beta)\} + N_1 \{nr - (\gamma_1 - \gamma)\} = 0.$$

For L_1, M_1, N_1, substitute these values and arrange the expressions in terms of λ, μ, ν; and putting for shortness

$$A_1 = n_1 m - m_1 n, \quad B_1 l = l_1 n - n_1 l, \quad C_1 = m_1 l - l_1 m$$
$$a_1 = m_1 (\gamma_1 - \gamma) - n_1 (\beta_1 - \beta), \quad b_1 = n_1 (a_1 - a) - l_1 (\gamma_1 - \gamma),$$
$$c_1 = l_1 (\beta_1 - \beta) - m_1 (a_1 - a),$$

we obtain the equation

$$(A_1 r + a_1)\lambda + (B_1 r + b_1)\mu + (C_1 r + c_1)\nu = 0 \; \ldots . (8).$$

In like manner, from the other equations of (7) we have

$$(A_2 r + a_2)\lambda + (B_2 r + b_2)\mu + (C_2 r + c_2)\nu = 0 \; \ldots . (9),$$
$$(A_3 r + a_3)\lambda + (B_3 r + b_3)\mu + (C_3 r + c_3)\nu = 0 \; \ldots . (10).$$

These three equations, along with the condition

$$\lambda^2 + \mu^2 + \nu^2 = 1,$$

are sufficient to determine λ, μ, ν and r, and from r the values of x', y', z' are found by (6). The best way of proceeding is first to eliminate λ, μ, ν from (8), (9), (10) by cross multiplication, when we obtain an apparent cubic of the form

$$(A_1 r + a_1)\{(B_2 r + b_2)(C_3 r + c_3) - (B_3 r + b_3)(C_2 r + c_2)\} +$$
$$(A_2 r + a_2)\{(B_3 r + b_3)(C_1 r + c_1) - (B_1 r + b_1)(C_3 r + c_3)\} +$$
$$(A_3 r + a_3)\{(B_1 r + b_1)(C_2 r + c_2) - (B_2 r + b_2)(C_1 r + c_1)\} = 0.$$

But if the coefficient of r^3 be examined it will be found to be identically equal to zero, so that the cubic is reduced to a quadratic of the form

$$Pr^2 + Qr + R = 0.$$

This gives two values of r: if they are both possible, there are two corresponding sets of values for λ, μ, ν, x', y', z', or two lines which satisfy the conditions of the problem: if the two values of r are impossible, the problem is impossible. When the values of r are known, they are to be substituted successively in the equations (8), (9), (10), from which λ, μ, ν may be determined in the ordinary way; but the expressions are so long and complicated that it is useless to write them down here.

VIII.—ON THE KNIGHT'S MOVE AT CHESS.

By R. Moon, M.A., Fellow of Queens' College.

It is some time since Dr. Roget published in the *Philosophical Magazine* his solution of the problem of the "Knight at Chess," when the initial and terminal squares are given. It

so happened that, a short time previous to the appearance of Dr. Roget's paper, my attention was directed to the subject by a friend, to whom it had been suggested by the perusal of a German work; and the result of my attempts at that time was the discovery of a method of solving the problem, (freed indeed from Dr. Roget's restriction as to the terminal square,) not only in the case of the common board of eight squares to a side, but also when the number of squares to a side is twelve, sixteen, or any multiple of four. I have more recently found, that the same method is applicable whatever be the number of sides, provided they exceed four; and with this further reservation, that if the number of squares be odd, and the central square be black when the initial square be white, *one* square must be allowed to be left uncovered. In fact, it is easy to see, *à priori*, that this last must be granted as a postulate; since, in the case supposed, the number of black squares in the board exceeds the number of white by one; and, by the conditions of the problem, a white and a black square must be covered alternately.

d	*c*	*b*	*a*	*d*	*c*	*b*	*a*
b	*a*	*d*	*c*	*b*	*a*	*d*	*c*
c	*d*	*D*	*C*	*B*	*A*	*a*	*b*
a	*b*	*B*	*A*	*D*	*C*	*c*	*d*
d	*c*	*C*	*D*	*A*	*B*	*b*	*a*
b	*a*	*A*	*B*	*C*	*D*	*d*	*c*
c	*d*	*a*	*b*	*c*	*d*	*a*	*b*
a	*b*	*c*	*d*	*a*	*b*	*c*	*d*

Take the case of the common board of eight squares to a side, and consider it as made up of a central square of four places to a side and the annulus about it.

If the Knight be placed on any square of the annulus, and be confined to move in it, it will return to the place from which it set out, after describing what we shall call a circuit. Thus, in the figure, the places marked with the letter *a* constitute a circuit. If the piece be placed on any square of the annulus not included in the first circuit, it may be made to describe a second distinct from the first; and the whole

annulus may be divided into four such circuits, which are respectively made up of the places marked with the letters a, b, c, d, respectively. The central square may be divided into four similar circuits. Also, if we have a board of twelve squares to a side, we may consider it as made up of a central square of eight places to a side, (which again may be subdivided in the manner above indicated), and an external annulus, which may be divided into four circuits; and similarly if we have a board of sixteen or twenty squares to a side.

Now it will be found, if we confine ourselves to one particular annulus, that it will in general be impossible to pass from one circuit to another. But if after describing a circuit in any annulus, or in the central square (which we will call principal divisions), we wish to pass to any other principal division, it will be found that, when we are moving *from* the centre, we may always pass from a particular letter in one division to any of the three other letters in the other division: thus, in the figure, we may always pass from A to b, d, and c, and so on. On the other hand, if we are moving *towards* the centre, in passing from one division to another the change may always be effected from any particular letter in the one to its *opposite* in the other, as from a to D, or from b to C, &c. Hence it is easy to write down the order in which the circuits should succeed each other for the common chess-board. Thus, if the first place be an a, the cycle will be

$$a\,D\,b\,C\,d\,A\,c\,B,$$
$$\text{or}\quad a\,D\,c\,B\,d\,A\,b\,C.$$

It is not meant that these are the only cycles which would succeed, but they are such as cannot fail, and the method of obtaining them is obvious.

When the board consists of twelve or more squares, some additional considerations are requisite. In this case, in fact, it is obvious that, if the initial place be in the outer annulus, and after describing one circuit in that we proceed to describe another in the middle division, and a third in the central square, and so move backwards and forwards from the outermost to the innermost division, and *vice versâ*, describing only one circuit in each division each time it is entered, we shall have exhausted the middle division before we have exhausted the two others; at the same time that it is impossible to proceed from the inner to the outer division *per saltum*, or without having recourse to the intermediate one. To obviate this difficulty, it must be observed in the first place, that in the central square it is always possible to move from B or C to A or D; so that if we arrive in the central square on a B or a C, we may describe two circuits at once,

and it will be found that this may *always* be effected. In the next place, it will be found, that in the external annulus we may pass from the circuit of *b* or *c* to that of *a* or *d*, or *vice versâ*, and thus describe two circuits at once, provided that, when we have concluded the first circuit, we are sufficiently near to the corners of the board; and this may always be contrived as we move *down* from the centre.

Hence, when the number of squares to a side is greater than eight, the rule is, to move in the first place up to the centre, and to continue moving backwards and forwards, taking care to cover two circuits of the innermost and outermost divisions whenever those divisions are entered.

Thus far as to the case when the number of squares to a side is a multiple of four. If the board be one of six squares to a side, it may be divided into a central square, containing four places in all, and an annulus which may be divided into four circuits. A board of ten places to a side may be divided into a central square of six places to a side, and an external annulus likewise divisible into four circuits; and similarly when the number of squares to a side is eighteen, twenty-two, &c. In all these cases, when the number to a side is greater than six, the central square of six must be treated in all respects as the central square of four to a side, when the number of places in the side of the board is a multiple of four, that is, when once entered it must be exhausted one half. In other respects the method is the same in the two cases.

If the number of squares in the board be odd, the same principle of division obtains. We shall still have a central square, which will have either five or seven places to a side, as the case may be, and a series of annuli divisible into circuits. It will be found, however, that in this case each annulus will consist of only two circuits; by reason of which the process is much simplified.

The limits necessarily prescribed in a publication of this kind, do not admit of my detailing the method of exhausting the squares of five, six, and seven places to a side; but after what has been said, it is probable that these will not present much difficulty. For the same reason I shall not attempt to point out the order of the circuits; to do which, so as to meet every case, would lead us to great length. I shall merely remark in conclusion, that it would not be difficult to show how Dr. Roget's restriction as to the final square might be complied with, if it were worth while to do so.

[A very convenient practical solution of the general problem on the ordinary board, is given in a work, "*Indian Reminiscences*," by A. Addison. London: Edward Bull. 1837.—Ed. M. J.]

IX.—ON CERTAIN CASES OF GEOMETRICAL MAXIMA AND MINIMA.

We occasionally meet in Geometry with certain cases of maxima or minima, for which the ordinary analytical process appears to fail, though from geometrical considerations it is obvious that maxima or minima do exist. The explanation of this failure is not given in works on the Differential Calculus, and some notice of it here may be acceptable to our readers. The difficulty and its explanation will be best seen in an example, and none is better suited for the purpose than a question proposed in one of the papers of the Smith's prize examination for 1842. This was—To explain the cause of the failure of the ordinary method of finding maxima and minima, when applied to the problem of finding the greatest or least perpendicular from the focus on the tangent to an ellipse, the perpendicular being expressed in terms of the radius vector.

The usual expression for the perpendicular in terms of the radius vector is

$$p^2 = \frac{b^2 r}{2a - r};$$

and as p^2 will be a maximum or minimum when p is so, the ordinary rule for finding maxima and minima gives us

$$\frac{d}{dr}(p^2) = \frac{2ab^2}{(2a-r)^2} = 0.$$

Now this equation can be satisfied only by $r = \pm \infty$, values which are not admissible in this case; whereas we know from geometry that p is a minimum when $r = a(1-e)$, and a maximum when $r = a(1+e)$.

It would appear then that these two values are not given by the analytical process, and the cause of this exception is to be explained. In the general theory of maxima and minima, it is assumed that the independent variable may receive all possible values; whereas in the present case r is limited to those values which are found by assigning all possible values to θ in the expression

$$r = \frac{a(1-e^2)}{1 - e\cos\theta}:$$

in other words, r is not absolutely independent. Now r being a function of another variable, admits itself of maximum and minimum values; and these are the values for which p is a maximum or minimum. The cause of the failure may therefore be thus exhibited: the equation

$$d(p^2) = \frac{2ab^2 dr}{(2a-r)^2} = 0$$

is satisfied by $dr=0$, that is, by making r a maximum or minimum. Hence generally, if we wish to find the maximum and minimum values of $y=f(x)$, we must consider, not only the values of x which satisfy the equation $\frac{dy}{dx}=0$, but also the maximum and minimum values of x itself.

In the 7th vol. of *Liouville's Journal*, p. 163, there is given a similar case of failure of the analytical process in the problem—To draw the shortest or longest line to a circle from a point without it. If we take the line passing through the centre of the circle, and the given point O as the axis of x, and call a the distance of the point from the centre, c the radius of the circle, x the co-ordinate of any point P in the circle measured from the centre, we shall have

$$OP^2 = a^2 + c^2 - 2ax, \text{ a max. or min.};$$

from which the usual process would give

$$\frac{d}{dx}(OP^2) = -2a = 0,$$

a nugatory result. In this case x is a maximum or minimum, while OP is a minimum or maximum, and therefore the equation to be satisfied is

$$d\,.\,(OP^2) = -2adx = 0,$$

which is satisfied by $dx=0$.

In this example the difficulty may be avoided by taking our co-ordinates generally, so that x shall not be a maximum or minimum when OP is so. We shall then have, calling a and b the co-ordinates of the centre of the circle, the other quantities as before,

$$OP^2 = a^2 + b^2 + c^2 - 2b\sqrt{(c^2 - x^2)} - 2ax = \text{max.};$$

whence, by the usual process,

$$\frac{bx}{\sqrt{(c^2 - x^2)}} = a,$$

$$\text{and} \qquad x = \pm \frac{ac}{\sqrt{(a^2 + b^2)}},$$

giving the two values of x, which will solve the problem.

A very instructive example will be found in the problem—To find those conjugate diameters in an ellipse of which the sum is a maximum or a minimum. If r and r_1 be any two conjugate diameters, a, b the axes, we have

$$r + r_1 = \text{maximum or minimum},$$

$$r^2 + r_1^2 = a^2 + b^2 = c^2 \text{ suppose},$$

so that $\qquad r + \sqrt{(c^2 - r^2)} = \text{maximum or minimum}.$

From this we have

$$dr\left\{1 - \frac{r}{\sqrt{(c^2 - r^2)}}\right\} = 0.$$

This equation is satisfied either by

$$r = \sqrt{(c^2 - r^2)}, \quad \text{i.e. by } r = \frac{c}{\sqrt{2}} = r_1,$$

or by $dr = 0$, which involves $dr_1 = 0$.

The former of these results gives the equal conjugate diameters, the sum of which is, as we know, a maximum. The latter result implies that both r and r_1 are maxima or minima, or that they are the principal axes, the sum of which is a minimum. By a different method we might have obtained the minimum instead of the maximum value of $r + r_1$, by the usual process for determining maxima and minima. For since $r^2 + r_1^2 = a^2 + b^2$ and $rr_1 \sin\theta = ab$, θ being the angle between the axes, we have

$$(r + r_1)^2 = a^2 + b^2 + \frac{2ab}{\sin\theta},$$

and hence $$\frac{d}{d\theta}(r + r_1)^2 = -\frac{2ab\cos\theta}{(\sin\theta)^2} = 0.$$

This is satisfied by $\cos\theta = 0$ or $\theta = \frac{1}{2}\pi$, implying that r and r_1 are the principal axes. In this case the maximum value of $r + r_1$ is given by $d\theta = 0$, since the equal conjugate diameters are those which make the greatest angle with each other.

G.

X.—ON THE SOLUTION OF CERTAIN FUNCTIONAL EQUATIONS.

By D. F. Gregory, M.A. Fellow of Trinity College.

In the fourteenth number of this Journal Mr. Leslie Ellis pointed out what appeared to him to be the essential difference between Functional Equations and those which are usually met with in the various branches of analysis. His idea is, that these classes of equations are distinguished by the *order* in which the operations are performed, so that, whereas in our ordinary equations the known operations are performed on those which are unknown, in functional equations the converse is the case, the unknown operations being performed on those which are known. As this view appears to me to be not only correct, but of very great importance for the proper understanding of the higher departments of analysis, I shall endeavour in the following pages to enforce and illustrate it.

On the preceding theory it is easy to see *why* the solution of functional equations must involve difficulties of a higher order than that of equations of the other class. For if we consider an equation as a series of operations performed on a subject, the operations being known and the subject unknown, the solution of the equation involves the finding of the subject, which may be done theoretically by undoing the operations which have been performed on it; that is, by effecting on the second side the inverse of the known operations on the first side. Thus, if we have the equation

$$\frac{dy}{dx} - ay = 0,$$

or, as we may write it,

$$\left(\frac{d}{dx} - a\right) \phi (x) = 0,$$

the object is to find the form of $\phi (x)$, which is readily done by performing the operation $\left(\frac{d}{dx} - a\right)^{-1}$ on both sides, when we have

$$\phi (x) = \left(\frac{d}{dx} - a\right)^{-1} 0 = C\epsilon^{ax}.$$

Here the whole difficulty lies in the performing of the inverse operation; and although practically the difficulty of doing so may be very great, yet it is a difficulty wholly different in *kind* from that which we meet with in trying to solve an equation in which the unknown operation is performed on that which is known. We have then no direct means of disengaging the unknown from the known operations, as the inverse of an unknown operation of course cannot be performed, and the known operation, being the subject, cannot be directly separated from the equation. Thus in the equation

$$\phi (ax) - \phi (x) = 0,$$

where the object is to determine the form of ϕ, we cannot as before write

$$\phi (ax - x) = 0,$$

since the form of ϕ is unknown, and we therefore cannot assume it to be subject to the distributive law; neither can we write

$$a\phi (x) - \phi (x) = 0,$$

since we cannot assume that ϕ and a are commutative operations.

The method which is followed for the solution of certain functional equations, is indicated by the process for the solu-

tion of linear equations in Finite Differences, which are in fact functional equations of a particular form. Thus the equation

$$u_{x+1} - a u_x = 0$$

might be written

$$\phi(x + 1) - a\phi(x) = 0,$$

in which the form of ϕ is to be determined.

Here the known operations are the subjects of the unknown, and we cannot directly disengage them; but we are enabled to do so by transforming the equation into one in which the unknown operation ϕ is the subject. For, assuming the operation D to be such that

$$D\phi(x) = \phi(x + 1),$$

we are able to investigate the laws of combination of this new symbol and its various properties, so as to make it a known operation. The equation then becomes

$$D\phi(x) - a\phi(x) = 0.$$

Now we can shew that D is a distributive symbol with respect to its subject, and that it is commutative with respect to a; we may therefore write the equation in the form

$$(D - a)\phi(x) = 0,$$

whence $$\phi(x) = (D - a)^{-1} 0.$$

For the complete solution, there remains only that we should know the inverse operations of $D - a$ or D, and these are found from the investigation of its direct action. The result, as we know, is

$$\phi(x) = Ca^x.$$

It is useless here to show how this theory may be extended to the solution of general linear equations in finite differences, as that has been sufficiently developed in other places: we shall therefore pass on to show that the same method may be applied to other functional equations.

Let us suppose ω to be any known operation performed on x, so that $\omega(x)$ is a known function of x, and let ϕ be an unknown operation; then the equation

$$\phi(\omega^n x) + a_1\phi(\omega^{n-1}x) + a_2\phi(\omega^{n-2}x) + \&c. + a_n\phi(x) = X,$$

in which $a_1, a_2, \ldots\ldots a_n$ are constants, and ϕ is a function to be determined, is a functional equation which bears a close analogy to the general linear equation in finite differences, and which may be solved by a similar process.

Let π be the symbol of an operation which, when performed on $\phi(x)$, converts it into $\phi(\omega x)$, so that

$$\pi\phi(x) = \phi(\omega x).$$

This symbol π possesses various properties in common with the symbol D and others, which are often used. Thus, since

$$\pi\pi\phi(x), \text{ or } \pi^2\phi(x) = \pi\phi(\omega x) = \phi(\omega\omega x) = \phi(\omega^2 x),$$

we see generally that when n is an integer

$$\pi^n\phi(x) = \phi(\omega^n x);$$

from which also it is easy to show that

$$\pi^m\pi^n\phi(x) = \pi^{m+n}\phi(x),$$

or the successive operations of π are subject to the index law. Again, we may consider π as a distributive function for

$$\pi\{f(x)+\phi(x)\} = f(\omega x)+\phi(\omega x) = \pi f(x) + \pi\phi(x).$$

Also, since π acts only on a function which involves x, it is commutative with respect to quantities not involving x; so that a being such a quantity,

$$a\pi = \pi a.$$

These are the laws which are used in applying the method of the separation of symbols to the solution of linear differential equations; and hence the same method may be applied to our functional equation. If we introduce into it the symbol π, it becomes

$$\pi^n\phi(x) + a_1\pi^{n-1}\phi(x) + \&c. + a_n\phi(x) = X,$$

which is no longer in a functional form, since the unknown operation ϕ is the subject of known operations. Separating the symbols of operation, we have

$$(\pi^n + a_1\pi^{n-1} + a_2\pi^{n-2} + \&c. + a_n)\,\phi(x) = X.$$

Now if $r_1, r_2, \ldots\ldots r_n$ be the roots of the equation,

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \&c. + a_n = 0,$$

the complex operation performed on $\phi(x)$ may, in consequence of the laws of combination given above, be decomposed into the simpler operations

$$(\pi - r_1)(\pi - r_2)\ldots\ldots(\pi - r_n)\,\phi(x) = X,$$

exactly as is done in linear differential equations. And if $N_1, N_2, \ldots\ldots N_n$ be the coefficients of the partial fractions arising from the decomposition of

$$\frac{1}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \&c. + a_n} = \frac{1}{(z-r_1)(z-r_2)\ldots\ldots(z-r_n)},$$

we have, by effecting the inverse operation of that on the left-hand side of the equation,

$$\phi(x) = N_1(\pi - r_1)^{-1}X + N_2(\pi - r_2)^{-1}X + \ldots\ldots + N_n(\pi - r_n)^{-1}X$$
$$+ (\pi - r_1)^{-1}0 + (\pi - r_2)^{-1}0 + \ldots\ldots + (\pi - r_n)^{-1}0.$$

The binomial operations in the first line may be expanded in integral powers of π, that is, according to successive performances of the known operation indicated by ω, and the results may therefore be assumed as known. But the operations in the second line must be developed in negative powers of π, implying the performance of inverse operations; the results of these must of course vary according to the nature of π or ω; and it is plain that any one of them is of the same form as that at which we should arrive in solving the equation

$$(\pi - r)\,\phi(x) = 0, \quad \text{or} \quad \pi\phi(x) - r\phi(x) = 0,$$

which is the simple functional equation

$$\phi(\omega x) - r\,\phi(x) = 0.$$

This may always be done, or supposed to be done, by Laplace's method, in which it is reduced to the solution of two equations of differences: one of these is always a linear equation of the first order, the other depends on the nature of the function represented by ω.

The preceding analysis shows us, that the solution of a certain class of functional equations may be reduced, exactly like linear equations in differentials and finite differences, to the determination of certain inverse operations, in the performance of which alone the difficulty of the solution lies: one or two examples may be of use in illustrating the theory.

Let $\omega(x) = mx$, m being a constant; and let the equation be of the second degree,

$$\phi(m^2x) + a\phi(mx) + b\phi(x) = x^n.$$

If α, β be the roots of $z^2 + az + b = 0$, this may by the preceding theory be put under the form

$$(\pi - \alpha)(\pi - \beta)\,\phi(x) = x^n,$$

where π is such that $\quad \pi\phi(x) = \phi(mx).$

Hence $\phi(x) = (\pi - \alpha)^{-1}(\pi - \beta)^{-1}x^n + (\pi - \alpha)^{-1}\,0 + (\pi - \beta)^{-1}\,0.$

The first term of the second side of the equation is easily determined: for since

$$\pi(x^n) = (mx)^n = m^n\,.\,x^n,$$

we may replace π by m^n, so that the term becomes

$$(m^n - \alpha)^{-1}\,(m^n - \beta)^{-1}\,x^n = \frac{x^n}{m^{2n} + am^n + b}.$$

There remains to determine the inverse operations, which are to be found from the solution of the functional equation

$$\phi_1(mx) - \alpha\phi_1(x) = 0 \,\ldots\ldots.(1).$$

Now, by Laplace's method, assume

$$x = u_s, \quad mx = u_{s+1},$$

so that $$u_{s+1} - mu_s = 0 \dots\dots\dots\dots (2),$$

an equation for determining u_s, which, being known, enables us to express z in terms of x. Equation (1) may be written as

$$\phi_1(u_{s+1}) - a\phi_1(u_s) = 0,$$

or simply $$v_{s+1} - av_s = 0 \dots\dots (3).$$

The integration of the equations (2) and (3), enables us to solve the given functional equation (3). The solution of (2), on the assumption that the arbitrary function is a constant, is

$$u_s = Cm^s = x;$$

whence, by changing the constant, we have

$$z = \frac{\log(cx)}{\log m}.$$

In like manner the solution of (3) is

$$b_s = Ca^s = C\varepsilon^{\frac{\log a}{\log m}\log(cx)} = C(cx)^{\frac{\log a}{\log m}} = \phi_1(x),$$

C being an arbitrary function of $\sin 2\pi z$ and $\cos 2\pi z$.

Similarly for β: hence the solution of the given functional equation is

$$\phi(x) = \frac{x^n}{m^{2n} + am^n + b} + C(cx)^{\frac{\log a}{\log m}} + C'(cx)^{\frac{\log \beta}{\log m}}.$$

Again, let $\omega(x) = x^n$, and the functional equation be

$$\phi(x^{n^2}) + a\phi(x^n) + b\phi(x) = \log x.$$

If we assume $x = u_s$, $x^n = u_{s+1}$, the solution of the preceding equation will depend on that of

$$u_{s+1} = u_s^n, \text{ and of } v_{s+1} - av_s = 0.$$

The integral of the former is

$$u_s = c^{n^s} = x;$$

whence, by a change of constant,

$$z = \frac{1}{\log n}\log\log(x^c).$$

The integral of the latter is

$$v_s = Ca^s = C\varepsilon^{\frac{\log a}{\log n}\log(\log x^c)} = C(\log x^c)^{\frac{\log a}{\log n}}.$$

Also, if $\phi(x^n) = \pi\phi(x)$, we have

$$\pi\log(x) = \log(x^n) = n\log x,$$

and therefore

$$(\pi^2 + a\pi + b)^{-1} \log x = \frac{\log x}{n^2 + an + b}.$$

Hence the solution of the proposed equation is

$$\phi(x) = \frac{\log x}{n^2 + an + b} + C(\log x^c)^{\frac{\log \alpha}{\log n}} + C_1 (\log x^c)^{\frac{\log \beta}{\log n}}.$$

If the function $\omega(x)$ be a periodic function of the n^{th} order, so that $\omega^n(x) = x$, $\omega^{n+1}(x) = \omega(x)$, &c., the result of such an operation as

$$(\pi - r)^{-1} f(x),$$

can be always readily determined. For, if we expand the binomial in ascending powers of π, it becomes

$$-\frac{1}{r}\left(1 + \frac{\pi}{r} + \frac{\pi^2}{r^2} + \&c. + \frac{\pi^{n-1}}{r^{n-1}} + \frac{\pi^n}{r^n} + \&c. + \frac{\pi^{2n}}{r^{2n}} + \&c.\right) f(x).$$

But as $\pi^n f(x) = f(\omega^n x) = f(x)$, this is equivalent to

$$-\frac{1}{r}\left(1 + \frac{1}{r^n} + \frac{1}{r^{2n}} + \&c.\right)\left(1 + \frac{\pi}{r} + \frac{\pi^2}{r^2} + \&c. + \frac{\pi^{n-1}}{r^{n-1}}\right) f(x)$$

$$= \frac{1}{1 - r^n}\{\pi^{n-1} + r\pi^{n-2} + \&c. + r^{n-2}\pi + r^{n-1}\} f(x)$$

$$= \frac{1}{1 - r^n}\{f(\omega^{n-1}x) + rf(\omega^{n-2}x) + \&c. + r^{n-2}f(\omega x) + r^{n-1}f(x)\}.$$

As an example, let us assume

$$\omega(x) = \frac{1 + x}{1 - x},$$

which is a periodic function of the fourth order, the successive results being

$$\omega^2(x) = -\frac{1}{x}, \quad \omega^3(x) = \frac{x - 1}{x + 1}, \quad \omega^4(x) = x.$$

Let the functional equation be

$$\phi\left(\frac{1 + x}{1 - x}\right) - a\phi(x) = x.$$

Then if $\quad x = u_z, \quad \dfrac{1 + x}{1 - x} = u_{z+1},$

$$u_{z+1}u_z - u_{z+1} + u_z + 1 = 0.$$

The solution of this is (Herschel's *Examples*, p. 34,)

$$u_z = \tan\left(C + \frac{\pi}{4} z\right) = x;$$

whence $\quad z = \dfrac{4}{\pi}(\tan^{-1} x - C).$

The solution of the equation

$$\phi\left(\frac{1+x}{1-x}\right) - a\phi(x) = 0,$$

is therefore

$$\psi(x) = Ca^s = Ca^{\frac{4}{\pi}(\tan^{-1} x - C)} = C_1 a^{\frac{4}{\pi}\tan^{-1} x},$$

by changing the arbitrary constant. Hence the proposed functional equation gives

$$\phi(x) = (\pi - a)^{-1} x + C_1 a^{\frac{4}{\pi}\tan^{-1} x}$$

$$= \frac{1}{1-a^4}\left(\frac{x-1}{x+1} - \frac{a}{x} + a^2\,\frac{1-x}{1+x} + a^3 x\right) + C_1 a^{\frac{4}{\pi}\tan^{-1} x}.$$

Again, let $\omega(x) = \dfrac{a^2}{x}$, a periodic function of the second order, and let the functional equation be

$$\phi\left(\frac{a^2}{x}\right) + \phi(x) = \epsilon^{nx}.$$

Then if $x = u_s$, $\dfrac{a^2}{x} = u_{s+1}$, we have

$$u_{s+1} u_s = a^2;$$

the integral of which is $u_s = aC^{(-1)^s} = x$;

$$\text{whence} \quad (-1)^s = \left(\log\frac{x}{a}\right)^c.$$

But the functional equation gives us

$$v_{s+1} + v_s = 0,$$

$$\text{whence} \quad v_s = C(-1)^s = C\left(\log\frac{x}{a}\right)^c.$$

$$\text{Hence} \quad \phi(x) = (\pi + a)^{-1}\epsilon^{nx} + C\left(\log\frac{x}{a}\right)^c$$

$$= \frac{1}{1-a^2}\left(\epsilon^{nx} - a\epsilon^{\frac{na^2}{x}}\right) + C\left(\log\frac{x}{a}\right)^c.$$

In conclusion, I may observe that this article does not pretend to give any new results, as the solution of Functional Equations of the kind here treated of is already known, (see Herschel's *Finite Differences*, p. 547): the object of it is merely to illustrate the theory before spoken of, and to show that a method which has been found useful in two departments of analysis may likewise be applied to simplify the processes of a more difficult branch.

XI.—MATHEMATICAL NOTES.

1. *Note on the Theory of Numbers.*—The following elegant demonstration of a known remarkable proposition in the Theory of Numbers, is given by M. E. Catalan in *Liouville's Journal*, tom. IV. p. 7.

If $\phi(n)$ represent the number of integers which are less than n and prime to it, and if d, d', d'', &c. be the divisors of a number N,

$$\phi(d) + \phi(d') + \phi(d'') + \&c. = N.$$

Let $N = a^\alpha b^\beta c^\gamma \ldots$, a, b, c, being the prime factors of N; then any divisor d may be represented by $a^i b^k c^l \ldots$, where the exponents $i, k, l \ldots$ vary from 0 to α, 0 to β, 0 to γ, &c. By a known theorem,

$$\phi(d) = d \cdot \frac{a-1}{a} \cdot \frac{b-1}{b} \cdot \frac{c-1}{c} \ldots$$

$$= a^{i-1}(a-1) \cdot b^{k-1}(b-1) \cdot c^{l-1}(c-1) \ldots,$$

each of the factors of the latter product being replaced by unity when the exponent involved in it is equal to -1. Now all the values of this product due to the different values which can be assigned to the exponents are given in the terms of the product of the quantities

$$1 + (a-1) + a(a-1) + a^2(a-1) + \ldots + a^{\alpha-1}(a-1),$$
$$1 + (b-1) + b(b-1) + b^2(b-1) + \ldots + b^{\beta-1}(b-1),$$
$$1 + (c-1) + c(c-1) + c^2(c-1) + \ldots + c^{\gamma-1}(c-1),$$
&c. &c.

Therefore, cancelling the terms which destroy each other,

$$\phi(d) + \phi(d') + \phi(d'') + \&c. = a^\alpha b^\beta c^\gamma \ldots = N.$$

2. *To shew that if*

$$l^2 + m^2 + n^2 = 1 \ldots\ldots (1),$$
$$l'^2 + m'^2 + n'^2 = 1 \ldots\ldots (2),$$
$$l''^2 + m''^2 + n''^2 = 1 \ldots\ldots (3),$$
$$l'l'' + m'm'' + n'n'' = 0 \ldots\ldots (4),$$
$$l''l + m''m + n''n = 0 \ldots\ldots (5),$$
$$ll' + mm' + nn' = 0 \ldots\ldots (6).$$

Then shall

$$l^2 + l'^2 + l''^2 = 1 \ldots\ldots (7),$$
$$m^2 + m'^2 + m''^2 = 1 \ldots\ldots (8),$$
$$n^2 + n'^2 + n''^2 = 1 \ldots\ldots (9),$$
$$mn + m'n' + m''n'' = 0 \ldots\ldots (10),$$
$$nl + n'l' + n''l'' = 0 \ldots\ldots (11),$$
$$lm + l'm' + l''m'' = 0 \ldots\ldots (12).$$

Putting (1) under the form

$$ll + mm + nn = 1, \text{ multiply by } m''n' - n''m',$$

$$(5) \ldots\ldots l''l + m''m + n''n = 0, \quad \ldots\ldots \quad m'n - n'm,$$

$$(6) \ldots\ldots l'l + m'm + n'n = 0, \quad \ldots\ldots \quad mn'' - nm''.$$

Then adding, we have, as in ordinary cross multiplication,

$$\{l(m''n' - n''m') + l''(m'n - n'm'') + l'(mn'' - nm'')\}\, l = m''n' - n''m',$$

or, for brevity, $\quad Sl = m''n' - n''m'.$

Hence, on account of the symmetry of S,

$$S = \frac{m''n' - n''m'}{l} = \frac{n''m - nm''}{l'} = \frac{nm' - n'm}{l''} \ldots\ldots (a),$$

and hence

$$l^2S + l'^2S + l''^2S = l(m''n' - m'n'') + l'(n''m - m''n) + l''(nm' - n'm),$$
$$= S,$$

$$\text{therefore} \quad l^2 + l'^2 + l''^2 = 1.$$

Similarly (8) and (9) may be proved. Again, from (a),

$$lmS + l'm'S + l''m''S = 0,$$

$$\text{therefore} \quad lm + l'm' + l''m'' = 0.$$

And in a similar manner (10) and (11) may be proved.

We have also, from (a),

$$(l^2 + l'^2 + l''^2)\, S^2 = (m''n' - n''m')^2 + (n''m - m''n)^2 + (nm' - n'm)^2$$
$$= (m^2 + m'^2 + m''^2)(n^2 + n'^2 + n''^2), \text{ on account of (10)},$$
$$= 1,$$

$$\text{therefore} \quad S^2 = 1.$$

Hence by (a) we can find l, l', l'', in terms of the other six quantities.

T.

CORRIGENDA.

Vol. III. p. 165, Equation (37), *for* L *read* $\mathcal{L}$.

.. p. 166, Equation (38), *for* $L - \frac{1}{kr}$, $L' - \frac{1}{kr'}$, *read* $\mathcal{L} - \frac{1}{kr}$, $\mathcal{L}' - \frac{1}{kr'}$.

.. p. 164-166, passim, *for* $\theta - \pi$, $\theta' - \pi'$, $\pi - \pi'$, *read* $\theta - \varpi$, $\theta' - \varpi'$, $\varpi - \varpi'$.

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I.—ON CERTAIN FORMULÆ MADE USE OF IN PHYSICAL ASTRONOMY.

SOME of the most important results of Physical Astronomy may be arrived at very readily by means of the following propositions, which are here demonstrated a little differently from the common method. In a future paper it is intended to apply the equations here obtained to the case of the Moon disturbed by the Sun, and of a planet disturbed by another planet.

1. *To obtain the polar formulæ for the forces which act upon a particle* (m) *moving in one plane.*

Let r be the radius vector of m at the time t, ω the angular velocity of r, and ϕ the angle which r makes with any fixed line (L) drawn in the plane of motion through the origin. Then, $\frac{dr}{dt}$ and $r\omega$ being the velocities of m along and perpendicular to r, the velocity of m parallel to (L) will be

$$\frac{dr}{dt}\cos\phi + r\omega\sin\phi\,;$$

(we suppose (L) to lie farther from the prime radius than r,) therefore the force on m parallel to (L) will be

$$\frac{d}{dt}\left(\frac{dr}{dt}\cos\phi + r\omega\sin\phi\right)$$

$$= \left(\frac{d^2r}{dt^2} - r\omega^2\right)\cos\phi + \left\{\frac{dr}{dt}\omega + \frac{d(r\omega)}{dt}\right\}\sin\phi \ldots . (1);$$

observing that $\frac{d\phi}{dt} = -\omega$, since (L) is fixed and farther from the prime radius than r.

A A

Now the position of (L) is arbitrary, therefore we may choose it so that at any time (t) it shall coincide with, or be perpendicular to r. Let (L) therefore coincide with r, i.e. let $\phi = 0$; then (1), which is now the force along r, becomes

$$\frac{d^2r}{dt^2} - r\omega^2.$$

Again, let (L) be perpendicular to r, i.e. let $\phi = \frac{\pi}{2}$; then (1), which now is the force perpendicular to r, becomes

$$\frac{dr}{dt}\omega + \frac{d(r\omega)}{dt} \qquad \text{or } \frac{1}{r}\frac{d(r^2\omega)}{dt}.$$

Hence, if at any time (t), P be the force on m along r, and Q that perpendicular to r, we have

$$\text{(A)} \left\{ \begin{array}{ll} \dfrac{d^2r}{dt^2} - r\omega^2 = P \;\ldots\ldots\ldots\ldots & (2). \\ \dfrac{1}{r}\dfrac{d(r^2\omega)}{dt} = Q \;\ldots\ldots\ldots\ldots & (3). \end{array} \right.$$

2. *If* T *be the force on* m *along its path, and* N *that perpendicular to it, to find* T *and* N.

Let v be the velocity of m, and ϕ the angle which (L) makes with the direction of v; then the velocity of m parallel to (L) is $v \cos\phi$, and therefore the force on m along (L) is

$$\frac{d(v\cos\phi)}{dt},$$

$$\text{or} \quad \frac{dv}{dt}\cos\phi - v\sin\phi\,\frac{d\phi}{dt} \;\ldots\ldots\ldots\ldots (4).$$

Now if ψ be the angle which the tangent to the path of m makes with the prime radius, we have $d\phi = -d\psi$ (we suppose (L) to make a greater angle with the prime radius than the tangent does), and $d\psi = \frac{ds}{\rho}$, ρ being the radius of curvature, and s the arc of the path of m:

hence $\frac{d\phi}{dt} = -\frac{1}{\rho}\frac{ds}{dt} = -\frac{v}{\rho}$, and therefore (4) becomes

$$\frac{dv}{dt}\cos\phi + \frac{v^2}{\rho}\sin\phi.$$

Hence putting $\phi = 0$ and $\frac{\pi}{2}$, as before, we find

$$\frac{dv}{dt} = T \qquad \frac{v^2}{\rho} = N.$$

3. The equations (A) in art. (1) may be transformed in the following manner.

Assume $$h = r^2 \frac{d\theta}{dt} \text{ and } \frac{1}{u} = r,$$

then
$$\frac{dr}{dt} = \frac{d}{d\theta}\left(\frac{1}{u}\right)\frac{d\theta}{dt} = -\frac{du}{d\theta}\frac{1}{u^2}\frac{d\theta}{dt}$$
$$= -\frac{du}{d\theta} h;$$
$$\therefore \frac{d^2r}{dt^2} = -\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} h - \frac{du}{d\theta}\frac{dh}{dt}$$
$$= -\frac{d^2u}{d\theta^2} h^2u^2 - \frac{du}{d\theta}\frac{Q}{u};$$
since $\dfrac{dh}{dt} = \dfrac{Q}{u}$ by (3).

Hence (2) becomes
$$P = -\frac{d^2u}{d\theta^2} h^2u^2 - \frac{du}{d\theta}\frac{Q}{u} - h^2u^3;$$
or
$$\frac{d^2u}{d\theta^2} + u + \frac{du}{d\theta}\frac{Q}{h^2u^2} + \frac{P}{h^2u^2} = 0 \quad \ldots\ldots (5).$$

Also, since $$\frac{dh}{dt} = \frac{dh}{d\theta}\frac{d\theta}{dt} = \frac{dh}{d\theta} hu^2,$$

(3) becomes $$h\frac{dh}{d\theta} = \frac{Q}{u^3};$$

and therefore $$h^2 = 2\int \frac{Q}{u^3}\, d\theta \quad \ldots\ldots\ldots\ldots (6).$$

(5) and (6) are the two equations made use of in the Lunar theory: the value of h^2 given in (6) is usually substituted in (5).

4. For a body acted on by a central force varying inversely as the square of the distance, we have
$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{\mu}{r^2} \ldots\ldots\ldots\ldots (1),$$
$$\frac{dh}{dt} = 0;$$
and hence h = constant.

Also, putting $r = \dfrac{1}{u}$, we have
$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta},$$
$$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2};$$

therefore (1) becomes

$$-h^2u^2\frac{d^2u}{d\theta^2}-h^2u^3=-\mu u^2,$$

$$\text{or } \frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2},$$

$$\text{whence} \qquad u=\frac{\mu}{h^2}+A\cos(\theta+B),$$

which shews that the orbit is a conic section, the pole being the focus.

5. To determine this conic section completely when the circumstances of projection are given, we have from (1), multiplying by $\frac{dr}{dt}$, and integrating,

$$\left(\frac{dr}{dt}\right)^2+\frac{h^2}{r^2}=\frac{2\mu}{r}+C \;\ldots\ldots\ldots\ldots (2).$$

$$\text{Now} \quad \left(\frac{dr}{dt}\right)^2+\frac{h^2}{r^2}=\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2=\left(\frac{ds}{dt}\right)^2=v^2;$$

$$\text{therefore} \qquad v^2=\frac{2\mu}{r}+C \;\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

When the particle is at either apse, $\frac{dr}{dt}=0$, and therefore by (2)

$$\frac{h^2}{r^2}=\frac{2\mu}{r}+C;$$

$$\text{or} \qquad r^2+\frac{2\mu}{C}r-\frac{h^2}{C}.$$

Now the roots of this equation are $a(1-e)$ and $a(1+e)$, therefore $-\frac{2\mu}{C}=$ their sum $=2a$, and therefore $C=-\frac{\mu}{e}$, and $-\frac{h^2}{C}=$ their product $=a^2(1-e^2)$, and $\therefore h^2=a\mu(1-e^2)$.

Hence we have

$$v^2=\frac{2\mu}{r}-\frac{\mu}{a} \;\ldots\ldots\ldots\ldots\ldots\ldots (\alpha),$$

$$h^2=a\mu(1-e^2) \;\ldots\ldots\ldots\ldots\ldots (\beta),$$

$$\text{and} \qquad r=\frac{a(1-e^2)}{1+e\cos(\theta-\varpi)} \;\ldots\ldots (\gamma).$$

Let u, c, and β, be the velocity, distance, and angle of projection; and take c to be the prime radius; then, when

$\theta = 0$, $r = c$, $v = u$: also we have $h = uc \sin \beta$. Hence, (α), (β), (γ), give

$$\frac{1}{a} = \frac{2\mu}{c} - u^2,$$

$$e^2 = 1 - \frac{u^2c^2 \sin^2 \beta}{a\mu},$$

$$\cos \pi = \frac{a(1 - e^2) - c}{ec},$$

which equations give a, e, and π.

6. A particle m moves anyhow in space: if P be the force on it along its radius vector r, Q that perpendicular to r in the plane of the orbit of m (*i.e.* the plane containing r and the direction of the motion of m at any time), and S the force perpendicular to this plane; *it is required to determine expressions for P, Q, and S.*

Let ω, as before, be the angular velocity of r, and η the angular velocity of the orbit round r. Let any fixed line (L) make an angle ϕ with r, and let the plane containing L and r make an angle ψ with the plane of the orbit. Then, the velocities of m being $\frac{dr}{dt}$ and $r\omega$, and the resolved part of $r\omega$ in the plane (Lr) being $r\omega \cos \psi$, the velocity parallel to (L) will be

$$\frac{dr}{dt} \cos \phi + r\omega \cos \psi \sin \phi,$$

and therefore the force on m, parallel to L, will be

$$\left(\frac{d^2r}{dt^2} + r\omega \frac{d\phi}{dt} \cos \psi\right) \cos \phi$$
$$+ \left(-\frac{dr}{dt}\frac{d\phi}{dt} + \frac{d(r\omega)}{dt} \cos \psi\right) \sin \phi - r\omega \sin \psi \sin \phi \frac{d\psi}{dt};$$

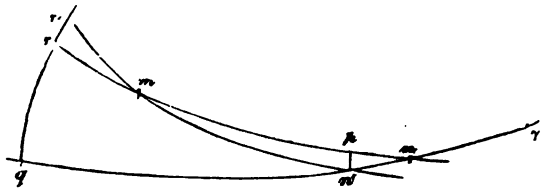
and this expression, if we put $\psi = 0$, $\phi = 0$, becomes P; if $\psi = 0$, $\phi = \frac{\pi}{2}$, it becomes Q; and if $\psi = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$, it becomes S. Moreover if $\psi = \frac{\pi}{2}$, $\frac{d\phi}{dt} = 0$, and $\frac{d\psi}{dt} = -\eta$, and if $\psi = 0$, $\frac{d\phi}{dt} = -\omega$. Hence we have

$$\frac{d^2r}{dt^2} - r\omega^2 = P,$$

$$\frac{1}{r}\frac{d(r^2\omega)}{dt} = Q,$$

$$r\omega\eta = S.$$

7. Let $\gamma n q$ (see figure) be the plane of the ecliptic, $n\ m$ the orbit of m; let $\theta = \gamma n + nm$, i.e. the longitude of m on its



orbit, and $\nu = \gamma n$ the longitude of the node. Then ωdt, the small angle described by r in the time dt, will be equal to the increment of mn + the motion of n in the time dt resolved along mn; i.e.

$$\omega dt = d(\theta - \nu) + d\nu \cos \iota,$$

ι being the inclination of the orbit to the ecliptic, therefore

$$\omega = \frac{d\theta}{dt} - 2\sin^2\frac{\iota}{2}\,\frac{d\nu}{dt}.$$

8. Let mn' be a consecutive position of the plane of the orbit, draw $n'p$ perpendicular to mn, and qrr' perpendicular to γnq, qn' being $\frac{1}{2}\pi$; then the angle $pmn' = \eta dt$, $n'n = d\nu$, and $rr' = d\iota$; therefore

$$d\nu = \frac{pn'}{\sin \iota} = \frac{\eta dt \sin(\theta - \nu)}{\sin \iota},$$

$$d\iota = \eta dt \sin mr = \eta dt \cos(\theta - \nu);$$

$$\therefore\quad \frac{d\nu}{dt} = \frac{\sin(\theta - \nu)}{\sin \iota}\,.\,\eta, \quad \frac{d\iota}{dt} = \cos(\theta - \nu)\,.\,\eta.$$

9. Hence, if we put $h = r^2\omega = r^2\left(\frac{d\theta}{dt} - 2\sin^2\frac{\iota}{2}\,\frac{d\nu}{dt}\right)$ we have the following general equations to determine the quantities r, θ, ν, ι,

$$\left.\begin{aligned}
&\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = P,\\
&\frac{dh}{dt} = Qr,\\
&\frac{d\nu}{dt} = \frac{Sr}{h}\,\frac{\sin(\theta - \nu)}{\sin \iota},\\
&\frac{d\iota}{dt} = \frac{Sr}{h}\cos(\theta - \nu),\\
&\text{where}\quad h = r^2\left(\frac{d\theta}{dt} - 2\sin^2\frac{\iota}{2}\,\frac{d\nu}{dt}\right),
\end{aligned}\right\}$$

r is the radius vector of m, θ is the longitude of m on its

orbit, ν the longitude of the node, and ι the inclination of the orbit.

10. Let M, m, m' be a free system of three mutually attracting bodies: then we may suppose one of them (M) to become fixed, provided we apply forces equal and opposite to those which act on M to the whole system. Let r be the distance Mm, $r'=Mm'$, $r_1=mm'$, then the forces which act on m will be

$$\frac{M}{r^2} \text{ along } mM, \qquad \frac{m'}{r_{\prime}^2} \text{ along } mm',$$

$$-\frac{m}{r^2} \text{ along } mM, \qquad -\frac{m'}{r'^2} \text{ parallel to } Mm'.$$

("Along Mm" means from M towards m, "along mM" means from m towards M.)

Now let Pm $(=s)$ be any curve (not necessarily in the plane Mmm'); then, if F be the force which acts on m along ds, we have

$$F = -\frac{M+m}{r^2}\cos(r, ds) - \frac{m'}{r_{\prime}^2}\cos(r_{\prime}, ds) - \frac{m'}{r'^2}\cos(r', ds).$$

$$\text{Now} \quad \cos(r, ds) = \frac{dr}{ds}, \qquad \cos(r_{\prime}, ds) = \frac{dr_{\prime}}{ds},$$

$$\text{and} \quad \cos(r', ds) = \frac{\text{projection of } ds \text{ on } Mm'}{ds};$$

$$= \frac{d\{r\cos(r, r')\}}{ds}.$$

$$\text{Hence} \quad F = -\frac{M+m}{r^2}\frac{dr}{ds} - \frac{m'}{r_{\prime}^2}\frac{dr_{\prime}}{ds} - \frac{m'}{r'^2}\frac{d\{r\cos(r, r')\}}{ds},$$

$$= \frac{d}{ds}\left(\frac{M+m}{r} + \frac{m'}{r_{\prime}} - \frac{m'r\cos(r, r')}{r'^2}\right).$$

And if we assume

$$M + m = \mu, \text{ and } R = m'\left\{\frac{1}{r_{\prime}} - \frac{r}{r'^2}\cos(r, r')\right\},$$

we have
$$F = \frac{d}{ds}\left(\frac{\mu}{r} + R\right);$$

In this differentiation we are to suppose m' fixed, and m to be displaced along s.

11. Hence we may find the forces P, Q, S, which act on m, (see Art. 6) as follows.

1st. Let s coincide with r, then $ds = dr$ and $F = P$; therefore

$$P = -\frac{\mu}{r^2} + \frac{dR}{dr}.$$

2ndly. Let s meet r at right angles and lie in the plane of the orbit of m; then $ds = rd\theta$, r does not vary, and $F = Q$; therefore

$$Q = \frac{dR}{rd\theta}.$$

3rdly. Let s meet r at right angles, and be perpendicular to the plane of m's orbit; then $ds = r\sin(\theta - \nu)\,d\iota$, r does not vary, and $F = S$; therefore

$$S = \frac{1}{r\sin(\theta - \nu)}\frac{dR}{d\iota}.$$

It is important to remember, that in these formulæ, $\frac{dR}{dr}$ is the differential coefficient of R on the supposition that θ, ν, ι, are constant, $\frac{dR}{d\theta}$ on the supposition that r, ν, ι, are constant, and $\frac{dR}{d\iota}$ on the supposition that r, θ, ν, are constant.

12. Hence we have the following equations to determine the motion of M:

$$\left.\begin{aligned} &\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{\mu}{r^2} + \frac{dR}{dr}, \\ &\frac{dh}{dt} = \frac{dR}{d\theta}, \\ &\frac{d\nu}{dt} = \frac{1}{h\sin\iota}\frac{dR}{d\iota}, \\ &\frac{d\iota}{dt} = \frac{1}{h}\cot(\theta - \nu)\frac{dR}{d\iota}, \\ \text{where}\quad &h = r^2\left(\frac{d\theta}{dt} - 2\sin^2\frac{\iota}{2}\frac{d\nu}{dt}\right). \end{aligned}\right\}$$

13. If we suppose s to coincide with the actual path of m, $ds = vdt$, and by Art. (2) we have

$$F(= T) = \frac{1}{v}\left(-\frac{\mu}{r^2}\frac{dr}{dt} + \frac{dR}{dt}\right),$$

and hence
$$v\frac{dv}{dt} = -\frac{\mu}{r^2}\frac{dr}{dt} + \frac{dR}{dt};$$

therefore
$$v^2 = \frac{2\mu}{r} + \int\frac{dR}{dt}\,dt,$$

$\frac{dR}{dt}$ is the differential of R with respect to t, supposing that r and θ alone are functions of t.

14. To find R, let λ and λ' be the latitudes of m and m', and $\theta_1\, \theta'_1$ the longitudes of m and m' measured on the ecliptic; then

$$\cos(r, r') = \sin\lambda \sin\lambda' + \cos\lambda \cos\lambda' \cos(\theta_1' - \theta_1),$$

$$r_1^2 = r^2 + r'^2 - 2rr' \cos(r, r'),$$

$$R = m' \left\{\frac{1}{r_1} - \frac{r}{r'^2} \cos(r, r')\right\}$$

$$\tan(\theta_1 - \nu) = \cos\iota \tan(\theta - \nu) \quad \tan(\theta_1' - \nu') = \cos\iota' \tan(\theta' - \nu')$$

$$\sin\lambda = \sin\iota \sin(\theta - \nu) \quad \sin\lambda' = \sin\iota' \sin(\theta' - \nu').$$

M. O. B.

II.—NOTE ON VIBRATING CORDS.

The equilibrium of a stretched cord is disturbed between the limits $x = l$ and $x = -l$, in such a manner as to cause a pulse, whose length is $2l$, to run along the cord in the positive direction: to find the portion of the resulting disturbance at any future time t, which is due to the disturbance of each element of the cord between $x = -l$ and $x = l$.

Let x and x' be the distances of two points, P and P', in the cord from the origin, the latter being between the limits $-l$ and l. Then $A \cos p\,(x - x' - at)$ satisfies the differential equation of the motion of the cord, and represents a series of waves, which we may suppose to originate from the point P', proceeding in the positive direction; and therefore, taking the sum of an infinite number of such systems, we find

$$\int_{-\infty}^{\infty} A \cos p\,(x - x' - at)\, dp,$$

for a disturbance proceeding in the positive direction from P'. But, if $A = 1$, the value of this integral is 0, except when

$$x - x' = at;$$

and it therefore represents an infinitely short pulse, leaving P', when $t = 0$. The total disturbance resulting from such pulse leaving every point P', between $x' = -l$ and $x' = l$, is

$$\int_{-l}^{l} Fx'dx' \int_{-\infty}^{\infty} \cos p\,(x - x' - at)\, dp;$$

and that this may represent the result of the given disturbance, it must be equal to the given initial disturbance, which we shall call fx', when $t = 0$. Hence, to determine Fx', we have

$$\int_{-l}^{l} Fx'dx' \int_{-\infty}^{\infty} \cos p\,(x - x')\, dp = fx'.$$

Hence, by Fourier's theorem, $Fx' = \frac{1}{\pi} fx'$, when x' is between $-l$ and l, and $Fx' = 0$, when x' is not between these limits. The part of the resulting disturbance, due to the point P', is therefore

$$\frac{1}{\pi} fx' dx' \int_{-\infty}^{\infty} \cos p (x - x' - at)\, dp.$$

The known result of a pulse in a vibrating cord, as it exists at any given period of time, has been thus expressed as the sum of a number of disturbances due to each part of the original pulse. The same method might possibly be applied to find the unknown result of an element of the surface of a pulse propagated in an elastic medium, such as air, or the ether which produces light, the disturbance of the element being supposed to diverge outwards in every direction, so as to form a spherical wave.

III.—ON THE INTEGRATION OF CERTAIN EQUATIONS OF FINITE DIFFERENCES.

By B. Bronwin.

The methods which I have employed in the integration of certain differential equations in two papers printed in this *Journal*, may be applied also to similar equations of finite differences. Without giving any general theory, I shall illustrate the subject in the present paper by a few examples.

Let $\quad (a + bx)\, \Delta^2 y_x + (h + kx)\, \Delta y_x - m y_x = 0,$

where $m = rk$, r being an integer. Taking the difference making $a_1 = a + b$, $h_1 = h + k + b$, $m_1 = m - k$, we have

$$(a_1 + bx)\, \Delta^3 y_x + (h_1 + kx)\, \Delta^2 y_x - m_1 \Delta y_x = 0.$$

Repeating this process r times, we at length find, since $m_r = 0$,

$$(a_r + bx)\, \Delta^{r+2} y_x + (h_r + kx)\, \Delta^{r+1} y_x = 0.$$

If $a_r = l$, $a_r - h_r = f$, $b - k = g$; the last equation may be put under the form

$$(l + bx)\, \Delta^{r+1} y_{x+1} - (f + gx)\, \Delta^{r+1} y_x = 0.$$

Making $\dfrac{f + gx}{l + bx} = \omega_x$, and integrating there results

$$\Delta^{r+1} y_x = CP\omega_{x-1}, \quad \Delta^r y_x = C\Sigma P\omega_{x-1} + C'.$$

Now, from the r equations obtained by taking successively the difference, we may eliminate Δy_x, $\Delta^2 y_x$, &c.; and shall have

$$y_x = M\Delta^r y_x + N\Delta^{r+1} y_x = C(M\Sigma P\omega_{x-1} + NP\omega_{x-1}) + C'M,$$

where C and C' are arbitraries, and M and N will be known

functions of x. This is the complete integral. M and N may sometimes be found by series precisely as in my first paper on differential equations.

Suppose $\Delta^2 y_x + kx\Delta y_x - 2ky_x = 0$; then $\omega_x = 1 - 2k - kx$, and $y_x = C\{(1 + kx + kx^2)\Sigma P\omega_{x-1} + xP\omega_{x-1}\} + C'(1 + kx + kx^2)$.

Let $x\Delta^2 y_x + kx\Delta y_x - 2ky_x = 0$. Here $\omega_x = \dfrac{2k+(k-1)x}{2+x}$,

and $y_x = C\{(2 + k + kx)x\Sigma(-1)^x P\omega_{x-1} + (1+x)x(-1)^x P\omega_{x-1}\}$
$+ C'(2 + k + kx)x$.

Let $(a + bx + cx^2)\Delta^2 y_x + (h + kx)\Delta y_x - my_x = 0$.
Taking the difference, making
$a_1 = a + b + c$, $b_1 = b + 2c$, $h_1 = h + k + b + c$, $k_1 = k + 2c$, $m_1 = m - k$;
we have $(a_1 + b_1 x + cx^2)\Delta^3 y_x + (h_1 + k_1 x)\Delta^2 y_x - m_1\Delta y_x = 0$.
If $m = rk + r(r-1)c$, repeating the process r times, we shall ultimately have

$$(a_r + b_r x + cx^2)\Delta^{r+2} y_x + (h_r + k_r x)\Delta^{r+1} y_x = 0.$$

Make $$\frac{a_r - h_r + (b_r - k_r)x + cx^2}{a_r + b_r x + cx^2} = \omega_x,$$

and the last equation may be put under the form

$$\Delta^{r+1} y_{x+1} - \omega_x \Delta^{r+1} y_x = 0,$$

which integrated gives

$$\Delta^{r+1} y_x = CP\omega_{x-1}, \quad \Delta^r y_x = C\Sigma P\omega_{x-1} + C'.$$

Eliminating Δy_x, $\Delta^2 y_x$, &c. as before, we find

$$y_x = M\Delta^r y_x + N\Delta^{r+1} y_x = C(M\Sigma P\omega_{x-1} + NP\omega_{x-1}) + C'M,$$

where M and N denote known functions of x, which result from the elimination.

As a more particular example, let

$$x(1+x)\Delta^2 y_x + k(1-x)\Delta y_x + ky_x = 0;$$

then $\quad (2 + 3x + x^2)\Delta^3 y_x + \{2 - (k-2)x\}\Delta^2 y_x = 0;$

or $\quad (2 + 3x + x^2)\Delta^2 y_{x+1} - \{(k+1)x + x^2\}\Delta^2 y_x = 0,$

$$\frac{(k+1)x + x^2}{2 + 3x + x^2} = \omega_x,$$

$$\Delta^2 y_x = CP\omega_{x-1}, \quad \Delta y_x = C\Sigma P\omega_{x-1} + C',$$

and $y_x = C(1-x) + C'\left\{(1-x)\Sigma P\omega_{x-1} + \dfrac{1}{k}(x + x^2)P\omega_{x-1}\right\}$.

In the preceding examples the coefficient of the last term is constant, but it is not necessary that it should be so. Thus the equation $\quad A_x\Delta^2 y_x + (a - x)B_x\Delta y_x + B_x y_x = 0$

is integrable after taking the difference once, the last term of the result vanishing when y_x is eliminated.

We shall now give a few examples where successive integration must be employed.

Let $(a+bx)\Delta^2 y_x + (h+kx)\Delta y_x + my_x = 0,\ m = rk.$

Integrating each term, and making

$$a_1 = a - b,\ h_1 = h - b - k,\ m_1 = m - k,$$

we have $(a_1 + bx)\Delta y_x + (h_1 + kx) y_x + m_1 \Sigma y_x + C = 0.$

Or if $\Sigma y_x + \dfrac{C}{m_1} = y'_x$, by substitution the last equation becomes

$$(a_1 + bx)\ \Delta^2 y'_x + (h_1 + kx)\ \Delta y'_x + m_1 y_x = 0.$$

Repeating this process r times, we shall at last find

$$(a_r + bx)\ \Delta^{r-1} y_x + (h_r + kx)\ y_x^{r-1} + C = 0,$$

which is equivalent to

$$y_{x+1}^{r-1} - \omega_x y_x^{r-1} + \frac{C}{a_r + bx} = 0, \quad \omega_x = \frac{a_r - h_r + (b-k)\,x}{a_r + bx}.$$

Thus y_x^{r-1} is known; from which we easily find in succession y_x^{r-2}, y_x^{r-3}, &c. and y_x, merely by the operation of differencing. And as y_x will contain two arbitraries, it will be the complete integral.

As a more particular example, let

$$x\Delta^2 y_x + kx\Delta y_x + 2ky_x = 0, \text{ and make } y'_x = \Sigma y_x + \frac{C}{k};$$

then we find successively

$$(x-1)\ \Delta y_x + \{k(x-1) - 1\}\ y_x + k\Sigma y_x + C = 0;$$
$$(x-1)\ \Delta^2 y'_x + \{k(x-1) - 1\}\ \Delta y'_x + ky'_x = 0;$$
$$(x-2)\ \Delta y'_x + \{k(x-2) - 2\}\ y'_x = C,$$

or if $$\omega_x = \frac{x}{x-2} - k,\ y'_{x+1} - \omega_x y'_x = \frac{C}{x-2};$$

therefore $$y'_x = CP\omega_{x-1}\Sigma \frac{1}{(x-2)P\omega_x} + C'P\omega_{x-1},$$

and consequently

$$y_x = C\Delta\left\{P\omega_{x-1}\Sigma \frac{1}{(x-2)\ P\omega_x}\right\} + C'\Delta P\omega_{x-1}.$$

Next let $(a + bx + cx^2)\ \Delta^2 y_x + (h + kx)\ \Delta y_x + my_x = 0.$

Integrating every term, there results

$$(a_1 + b_1 x + cx^2)\ \Delta y_x + (h_1 + k_1 x)\ y_x + m_1 \Sigma y_x + C = 0;$$
$$a_1 = a - b + c,\ b_1 = b - 2c,\ h_1 = h - b + 3c - k,\ k_1 = k - 2c,$$
$$m_1 = m - k + 2c.$$

Make $$\Sigma y_x + \frac{C}{m_1} = y'_x;$$
then the last, by substitution, becomes
$$(a_1 + b_1 x + cx^2)\,\Delta^2 y'_x + (h_1 + k_1 x)\,\Delta y'_x + m_1 y'_x = 0.$$
If $m = rk - r(r+1)c$, we shall have, by repeating the integration r times,
$$(a_r + b_r x + cx^2)\,\Delta y_x^{r-1} + (h_r + k_r x)\,y_x^{r-1} + C = 0,$$
which is integrable, and hence the integral of the proposed is readily found.

The method of series will apply to equations of finite differences similarly as to differential equations, factorials supplying the place of powers.

If $$y_x = A_0 + A_1 x + A_2 x x_{-1} + A_3 x x_{-1} x_{-2} + ;$$
$$\Delta y_x = A_1 + 2A_2 x + 3A_3 x x_{-1} + ; \text{ and } \Delta^2 y_x = 2A_2 + 2.3A_3 x + ;$$
where $$x_{-1} = x - 1,\ x_{-2} = x - 2,\ \&c.$$
Now let $$(a + bx)\,\Delta^2 y_x - bx\Delta y_x + hy_x = 0.$$
The scale of this is
$$(n+2)(n+1)(a+nb)\,A_{n+2} + (h - nb)\,A_n = 0;$$
from which we readily find
$$y_x = A_0\left\{1 - \frac{h}{2a}xx_{-1} + \frac{h(h-2b)}{2.3.4a(a+2b)}xx_{-1}x_{-2}x_{-3} - \frac{h(h-2b)(h-4b)}{2.3.4.5.6a(a+2b)(a+4b)}x\ldots x_{-5} + \right\}$$
$$+A_1\left\{x - \frac{h-b}{2.3(a+b)}xx_{-1}x_{-2} + \frac{(h-b)(h-3b)}{2.3.4.5(a+b)(a+3b)}xx_{-1}x_{-2}x_{-3}x_{-4} - \right\},$$
the complete integral, having two arbitraries A_0 and A_1. But one of these series will fail when $a + ib = 0$, i an integer. In this case we find, making
$$n = 2 - \frac{a}{b},\ 4 - \frac{a}{b},\ \&c.;\ A_{4-\frac{a}{b}} = -\frac{b(h - 2b + a)}{2(a - 3b)(a - 4b)}A_{2-\frac{a}{b}},$$
$$A_{6-\frac{a}{b}} = \frac{b^2(h - 2b + a)(h - 4b + a)}{2.4(a - 3b)(a - 4b)(a - 5b)(a - 6b)}A_{2-\frac{a}{b}},\ \&c.;$$
which will give a new series to supply the defective one, the first term of which is
$$A_{2-\frac{a}{b}}.\,xx_{-1}.\,\ldots\ldots\ldots\, x_{-1+\frac{a}{b}}.$$

We find for a descending series, making $a+h=k$ to abridge,
$$y_x = A_{\frac{k}{b}}.xx_{-1}.\ldots x_{-\frac{k}{b}+1}\left\{1 - \frac{h(h-b)(k-2b)}{2b^2(bx - h + b)}\right.$$
$$\left. + \frac{h(h-b)(h-2b)(h-3b)(k-2b)(k-4b)}{2\,.\,4b^4(bx - h + b)(bx - h + 2b)} - \right\}.$$

But this is only a particular integral, and we cannot find the other particular integral by a descending series.

Let $x(x+1)\Delta^2 y_x - kx\Delta y_x + my_x = 0$,

where $k = p + q - 1$, $m = pq$. Make $y_x = \Sigma A_n x x_{-1} \ldots x_{-n+1}$.
Then $\Delta y_x = \Sigma n A_n x x_{-1} \ldots x_{-n+2}$, $\Delta^2 y_x = \Sigma n n_{-1} A_n x x_{-1} \ldots x_{-n+3}$.

We find the scale to be

$$n_2 n_1^2 n A_{n+2} + n_1 n (2n - k) A_{n+1} + (p - n)(q - n) A_n = 0.$$

Here $n_1 = n + 1$, $n_2 = n + 2$, $n_{-1} = n - 1$, &c. to abridge.

Making $n = p$, $p - 1$, $p - 2$, &c., $A_{p+1} = 0$; and then $n = q$, $q - 1$, &c., we obtain

$$y_x = A_p x x_{-1} \ldots x_{-p+1} \cdot \left\{1 + \frac{p(p-1)}{x-p+1} + \frac{p(p-1)^2(p-2)}{2(x-p+1)(x-p+2)} + \text{\&c.}\right\}$$

$$+ A_q x x_{-1} \ldots x_{-q+1} \cdot \left\{1 + \frac{q(q-1)}{x-q+1} + \frac{q(q-1)^2(q-2)}{2(x-q+1)(x-q+2)} + \text{\&c.}\right\}$$

When p and q are not integers, the factorials $xx_{-1} \ldots x_{-p+1}$, $xx_{-1} \ldots x_{-q+1}$ can only be approximately calculated.

The above could be integrated by successively taking the difference when p or q is an integer, as the last term would vanish after p or q operations.

Making $n = 0, 1, 2$, &c., we might obtain two ascending series. But they would not terminate when p or q is integer. Possibly however, by changing the arbitraries for others, series that terminate in the case supposed might be obtained.

If $(a + bx)\Delta^2 y_x + (h + kx)\Delta y_x - rky_x = 0$, r a positive integer,

and $y_x = \Sigma A_n x x_{-1} \ldots x_{-n+1}$;

the scale is

$$n_2 n_1 (a + nb) A_{n+2} + n_1 (h + nb + nk) A_{n+1} + (n - r) k A_n = 0.$$

From this we find

$$A_{r-1} = r B_0 A_r, \quad A_{r-2} = r r_{-1} B_1 A_r, \quad A_{r-3} = r r_{-1} r_{-2} B_2 A_r, \text{ \&c.};$$

B_0, B_1, B_2, &c. being put for known quantities. We shall therefore have a particular integral in a descending series which will terminate. This will be the expression for $C'M$ in our first equation.

We may find the complete integral in two ascending series, having A_0 and A_1 for arbitraries; but neither of these series will terminate. This may seem a little paradoxical, but perhaps will admit of explanation. I shall not however attempt to give any, from fear of extending this paper to too great a length.

Let $$y_{x+2} + (a + bx)\, y_{x+1} + (h + kx)\, y_x = 0.$$
Make $y_x = m^x \sin(\pi x + \beta) z_x$, where π is 3.14159, and β an arbitrary constant. Substituting this quantity, the proposed becomes $$m^2 z_{x+2} - m(a + bx)\, z_{x+1} + (h + kx) z_x = 0,$$
or $$\Delta^2 z_x + (f - gx)\,\Delta z_x + l z_x = 0,$$
making $m = \dfrac{k}{b}$, $f = 2 - \dfrac{ab}{k}$, $g = \dfrac{b^2}{k}$, $l = 1 - \dfrac{ab}{k} + \dfrac{b^2 h}{k^2}$.

Here a particular integral suffices on account of the arbitrary β. And if l be a multiple of g positive or negative, we shall have one in finite terms. If we cannot have a series which terminates, we may assign any value to one of the two arbitraries, in order to simplify the resulting series.

Let $(a + bx)\, y_{x+2} + (f + gx)\, y_{x+1} + (h + kx)\, y_x = 0.$

The same substitution, as in the last example, will give
$$(a + bx)\,\Delta^2 z_x + (f_1 + g_1 x)\,\Delta z_x + k_1 z_x = 0.$$
$$2a - \frac{f}{m} = f_1, \quad 2b - \frac{g}{m} = g_1, \quad a - \frac{g}{m} + \frac{h}{m^2} = k_1,$$
and $$bm^2 - gm + k = 0.$$

The two values of m give two equations; a particular integral of either will suffice.

In this last example we might have made $y_x = m^x z_x$; and a particular integral of each of the resulting equations, if distinct, would give the complete integral.

Instead of forming the scale, we might proceed thus: make
$$y_x = A_0 + A_1 x + A_2 x(x - 1) +, \text{ or}$$
$$y_x = A_0 + A_1 \frac{x}{p + qx} + A_2 \frac{x(x-1)}{(p + qx)(p_1 + q_1 x)} +, \text{ or}$$
$$y_x = A_0 f_1(x) + A_1 x f_2(x) + A_2 x(x - 1) f_3(x) + ;$$
$f_1(x)$, $f_2(x)$, denoting any functions of x.

Make $x = 0, 1, 2$, &c., successively, and we have A_0, A_1, A_2, &c. in terms of y_0, y_1, y_2, &c. But the proposed equation gives y_2, y_3, &c. in terms of y_0, y_1. We have therefore the coefficients in terms of these last quantities.

From these assumed series we may find results commencing at any term, and forming either an ascending or descending series.

[*Note*—In my last paper, p. 182,
$$y = C\int_0^1 t^{q-1}\,dt(1-t)^{q-1}\cos(2rxt - rx) + C'\int_0^1 t^{q-1}\,dt(1-t)^{q-1}\sin(2rxt - rx),$$

is erroneously given as the complete integral of

$$x\frac{d^2y}{dx^2} + 2q\frac{dy}{dx} + r^2xy = 0.$$

The second integral vanishes between the limits.]

IV.—ON A CLASS OF DIFFERENTIAL EQUATIONS, AND ON THE LINES OF CURVATURE OF AN ELLIPSOID.

By A. Cayley, B.A. Fellow of Trinity College.

Consider the primitive equation

$$fx + gy + hz + \ldots = 0 \ldots\ldots\ldots\ldots (1),$$

between (n) variables x, y, z, the constants f, g, h being connected by the equation

$$H(f, g, h \ldots) = 0 \ldots\ldots\ldots\ldots (2),$$

H denoting a homogeneous function. Suppose that $f, g, h \ldots$ are determined by the conditions

$$fx_1 + gy_1 + hz_1 \ldots = 0 \ldots\ldots\ldots\ldots (3),$$
$$\cdots\cdots\cdots$$
$$fx_{n-2} + gy_{n-2} + hz_{n-2} \ldots = 0.$$

Then writing $$x = \begin{vmatrix} y, z & \ldots \\ y_1, z_1 & \ldots \\ \vdots & \\ y_{n-2} z_{n-2} & \ldots \end{vmatrix} \ldots\ldots\ldots\ldots (4).$$

With analogous expressions for $y, z \ldots$ the equations (3) give $f, g, h, \ldots$ proportional to $x, y, z, \ldots$ or eliminating $f, g, h \ldots$ by the equation (2),

$$H(x, y, z \ldots) = 0 \ldots\ldots\ldots\ldots (5).$$

Conversely the equation (5), which contains, in appearance, $n(n-2)$ arbitrary constants, is equivalent to the system (1) (2). And if H be a rational integral function of the order (r), the first side of the equation (5) is the product of r factors, each of them of the form given by the system (1), (2).

Now from the equation (1), we have the system

$$fx + gy + hz \ldots = 0 \ldots\ldots\ldots (6).$$
$$fdx + gdy + hdz \ldots = 0$$
$$\cdots\cdots\cdots$$
$$fd^{n-2}x + gd^{n-2}y + hd^{n-2}z \ldots = 0,$$

Or writing $$X = \begin{vmatrix} y, z & \ldots \\ dy, dz & \ldots \\ \vdots & \\ d^{n-2}y, d^{n-2}z & \ldots \end{vmatrix} \ldots\ldots\ldots\ldots (7)$$

With analogous expressions for $Y, Z \ldots$ from the equations (6), $f, g, h \ldots$ are proportional to $X, Y, Z \ldots$ or eliminating by (2)

$$H(X, Y, Z \ldots) = 0 \ldots\ldots\ldots\ldots (8).$$

Conversely the integral of the equation (8) of the order $(n-2)$ is given either by the system of equations (1), (2), in which it is evident the number of arbitrary constants is reduced to $(n-2)$; or, by the equation (5), which contains in appearance $n(n-2)$ arbitrary constants, but which we have seen is equivalent in reality to the system (1),(2).

Thus, with three variables, the integral of

$$H(ydz - zdy,\ zdx - xdz,\ xdy - ydx) = 0 \ldots (9)$$

may be expressed in the two forms

$$H(yz_1 - y_1 z,\ zx_1 - z_1 x,\ xy_1 - x_1 y) = 0 \ldots (10),$$

and

$$fx + gy + hz = 0 \ldots\ldots\ldots\ldots (11),$$

where

$$H(f, g, h) = 0 \ldots\ldots\ldots\ldots (12).$$

The above principles afford an elegant mode of integrating the differential equation for the lines of curvature of an ellipsoid. The equation in question is

$$(b^2 - c^2)\,xdydz + (c^2 - a^2)\,ydzdx + (a^2 - b^2)\,zdxdy = 0 \ldots (13),$$

where x, y, z are connected by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots\ldots (14);$$

writing

$$\frac{x^2}{a^2} = u,\ \frac{y^2}{b^2} = v,\ \frac{z^2}{c^2} = w \ldots\ldots\ldots\ldots (15),$$

these become

$$(b^2 - c^2)\,udvdw + (c^2 - a^2)\,vdwdu + (a^2 - b^2)\,wdudv = 0 \ldots (16),$$

$$u + v + w = 1 \ldots\ldots\ldots\ldots (17).$$

Multiplying by

$$-\{(vdu - udv)(wdv - vdw)(udw - wdu)\}^{-1},$$

the first of these becomes

$$\frac{-a^2du}{(vdu-udv)(udw-wdu)} - \frac{b^2dv}{(wdv-vdw)(vdu-udv)} - \frac{c^2dw}{(udw-wdu)(wdv-vdw)} = 0 \ldots (18);$$

but writing (17) and its derived equations under the form

$$u + (v + w) = 1 \ldots\ldots\ldots\ldots (19),$$

$$du + (dv + dw) = 0,$$

we deduce $\quad -du\,(v + w) + u\,.(dv + dw) = -du \ldots\ldots (20),$

i.e. $\quad -du = -(vdu - udv) + (udw - wdu) \ldots (21).$

and similarly

$$-dv = -(wdv - vdw) + (vdu - udv)$$

$$-dw = -(udw - wdu) + (wdv - vdw).$$

Substituting,

$$\frac{b^2-c^2}{wdv-vdw}+\frac{c^2-a^2}{udw-wdu}+\frac{a^2-b^2}{vdu-udv}=0 \ldots (22);$$

the integral of which may be written under either of the forms

$$\frac{b^2-c^2}{wv_1-vw_1}+\frac{c^2-a^2}{uw_1-wu_1}+\frac{a^2-b^2}{vu_1-v_1u}=0 \ldots\ldots (23).$$

where, on account of (17),

$$u_1+v_1+w_1=1 \ldots\ldots\ldots\ldots (24).$$

Or

$$fu+gv+hw=0 \ldots\ldots\ldots\ldots (25),$$

f, g, h being connected by

$$\frac{b^2-c^2}{f}+\frac{c^2-a^2}{g}+\frac{a^2-b^2}{h}=0 \ldots\ldots\ldots (26),$$

an equation which is satisfied identically by

$$f=\frac{b^2-c^2}{B^2-C^2},\ g=\frac{c^2-a^2}{C^2-A^2},\ h=\frac{a^2-b^2}{A^2-B^2} \ldots (27).$$

Restoring x, y, z, x_1, y_1, z_1 for u, v, w, u_1, v_1, w_1, the equations to a line of curvature passing through a given point x_1, y_1, z_1, on the ellipsoid, are the equation (14) and

$$\frac{(b^2-c^2)}{a^2(y_1^2z^2-y^2z_1^2)}+\frac{(c^2-a^2)}{b^2(z_1^2x^2-z^2x_1^2)}+\frac{(a^2-b^2)}{c^2(x_1^2y^2-x^2y_1^2)}=0 \ldots (28).$$

Or again, under a known form, the equation (14) and

$$\frac{(b^2-c^2)}{B^2-C^2}\cdot\frac{x^2}{a^2}+\frac{c^2-a^2}{C^2-A^2}\cdot\frac{y^2}{b^2}+\frac{a^2-b^2}{A^2-B^2}\cdot\frac{z^2}{c^2}=0 \ldots (29).$$

From the equations (14), (29) it is easy to prove the well known form

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1 \ldots\ldots\ldots (30);$$

in fact, multiplying (29) by m, and adding to (14), we have the equation (30), if the equations

$$\frac{1}{a^2}+m\cdot\frac{b^2-c^2}{B^2-C^2}\cdot\frac{1}{a^2}=\frac{1}{a^2+\theta} \ldots\ldots\ldots (31),$$

$$\frac{1}{b^2}+m\cdot\frac{c^2-a^2}{C^2-A^2}\cdot\frac{1}{b^2}=\frac{1}{b^2+\theta},$$

$$\frac{1}{c^2}+m\cdot\frac{a^2-b^2}{A^2-B^2}\cdot\frac{1}{c^2}=\frac{1}{(c^2+\theta)}$$

are satisfied.

But on reduction, these take the form

$$(B^2-C^2)\,\theta+(b^2-c^2)\,m\theta+ma^2(b^2-c^2)=0 \ldots (32).$$

$$(C^2-A^2)\,\theta+(c^2-a^2)\,m\theta+mb^2(c^2-a^2)=0,$$

$$(A^2-B^2)\,\theta+(a^2-b^2)\,m\theta+mc^2(a^2-b^2)=0.$$

And since, by adding, an identical equation is obtained, m and θ may be determined to satisfy these equations. The values of θ, m are

$$\theta = \frac{(a^2-b^2)(b^2-c^2)(c^2-a^2)}{a^2(B^2-C^2)+b^2(C^2-A^2)+c^2(A^2-B^2)} \ldots\ldots (33),$$

$$m = \frac{b^2c^2(B^2-C^2)+c^2a^2(C^2-A^2)+a^2b^2(A^2-B^2)}{(a^2-b^2)(b^2-c^2)(c^2-a^2)} \ldots (34).$$

V.—NOTE ON A PROBLEM IN DYNAMICS.

In the problem of finding the trajectory of a body under the action of a central force varying inversely as the square of the distance, when the circumstances of projection are given, it is usual to employ polar co-ordinates, either r and θ or p and r. The problem may, however, be solved quite as readily and more elegantly by adhering to rectilinear co-ordinates. To shew this take the equations of motion,

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{r^3} \ldots\ldots (1), \qquad \frac{d^2y}{dt^2} = -\frac{\mu y}{r^3} \ldots\ldots (2),$$

whence, as usual, we have

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h \ldots\ldots (3).$$

Multiplying (1) by (3), we have

$$h\frac{d^2x}{dt^2} = -\frac{\mu}{r^3}\left(x^2\frac{dy}{dt} - xy\frac{dx}{dt}\right)$$

$$= -\frac{\mu}{r^2}\left(r\frac{dy}{dt} - y\frac{dr}{dt}\right) = -\mu\frac{d}{dt}\left(\frac{y}{r}\right).$$

Therefore on integration

$$-h\frac{dx}{dt} = \mu\frac{y}{r} + g \ldots\ldots (4),$$

g being an arbitrary constant.

Similarly from (2) and (3), we have

$$h\frac{dy}{dt} = \mu\frac{x}{r} + f \ldots\ldots (5).$$

Multiply (4) by y, and (5) by x, and add, then

$$\mu r + fx + gy = h^2 \ldots\ldots (6).$$

From this it appears that r the radius vector is a rational and integral function of the co-ordinates of its extremity (x and y), and therefore this is the equation to a conic section, the focus of which is at the origin.

On comparing (6) with the general polar equation to the conic section,

$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\alpha)},$$

or $\quad r + e\cos\alpha x + e\sin\alpha y = a(1-e^2),$

we find $\quad a(1-e^2) = \frac{h^2}{\mu}, \quad e^2 = \frac{f^2+g^2}{\mu^2},$

$$a = \frac{\mu h^2}{\mu^2 - (f^2+g^2)}, \quad \tan\alpha = \frac{g}{f}.$$

To determine the velocity in the path, square and add (4) and (5), then

$$h^2v^2 = \mu^2 + \frac{2\mu}{r}(fx+gy) + a^2 + b^2,$$

But from (6) $\quad fx + gy = h^2 - \mu r,$

therefore $\quad h^2v^2 = h^2\frac{2\mu}{r} + f^2 + g^2 - \mu^2 \ldots\ldots (7).$

Now if V be the velocity of projection at the distance ρ, we have from (7),

$$h^2V^2 = h^2\frac{2\mu}{\rho} + f^2 + g^2 - \mu^2 \ldots\ldots (8),$$

whence, by subtracting, and dividing by h^2,

$$v^2 = V^2 + 2\mu\left(\frac{1}{r} - \frac{1}{\rho}\right) \ldots\ldots (9).$$

Again from (3), we have

$$\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right)\frac{ds}{dt} = h.$$

But if δ be the angle between ρ and the direction of projection,

$$x\frac{dy}{ds} - y\frac{dx}{ds} = \rho\sin\delta$$

at the point of projection: and as then we have $\frac{ds}{dt} = V$,

$$h = V\rho\sin\delta \ldots\ldots (10).$$

Then from (8) $\quad \frac{f^2+g^2-\mu^2}{h^2} = V^2 - \frac{2\mu}{\rho};$

and therefore $\quad a = \frac{\mu}{\frac{2\mu}{\rho} - V^2} \ldots\ldots\ldots\ldots (11).$

Also $\quad e^2 = 1 - \frac{V^2\rho^2\sin^2\delta}{\mu^2}\left(\frac{2\mu}{\rho} - V^2\right) \ldots . (12).$

As the species of conic section depends on whether e is less, equal to, or greater than unity, it appears that the path is an ellipse, a parabola, or a hyperbola, according as

$$\frac{2\mu}{\rho} > = \text{ or } < V^2,$$

and therefore the species of conic section is independent of the angle of projection.

To find the angle α, we have

$$\tan \alpha = \frac{g}{f}, \quad \text{or } \cos \alpha = \frac{f}{\sqrt{(f^2+g^2)}} = \frac{f}{\mu e}.$$

Now supposing the direction of projection to coincide with the axis of x, we have $x = \rho$ when $y = 0$, hence (6) gives us

$$f = \frac{h^2}{\rho} - \mu = V^2\rho \sin^2 \delta - \mu;$$

and therefore

$$\cos \alpha = \frac{V^2\rho \sin^2 \delta - \mu}{\mu e} = \frac{V^2\rho \sin^2 \delta - \mu}{\left\{\mu^2 - V^2\rho^2 \sin^2 \delta \left(\frac{2\mu}{\rho} - V^2\right)\right\}^{\frac{1}{2}}}.$$

If we combine (8) and (11) we get, as an expression for the velocity,
$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}.$$

G.

VI.—ON THE INTEGRATION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS BY THE METHODS OF MONGE AND LAGRANGE.

By W. Walton, M.A. Trinity College.

In Professor De Morgan's valuable work on the Differential and Integral Calculus (p. 722), the method of Monge for obtaining the first integrals of a partial differential equation of the second order and first degree, is objected to as probably from the nature of the case leading to results of insufficient generality. This method commences with assuming the coefficient of s, and consequently the left-hand member in the equation

$$Pdy\, dp + Rdx\, dq - Sdx\, dy = s\,(Pdy^2 - Qdx\, dy + Rdx^2),$$

or, as it may be written, $\sigma = s\alpha$, each equal to zero. The Professor observes that "in the equation $\sigma = s\alpha$ we have begun by presuming the existence of a solution which allows α to vanish, when of course σ vanishes. The solution we thus obtain may be the most general of its kind,—that is, of those which allow α and σ to vanish; but how do we ascertain that there are no solutions in which this is impossible? or how do

we know that there are not some in which when a vanishes, s necessarily becomes infinite?" The generality of Lagrange's integration of partial differential equations of the first order, the Professor establishes by à posteriori demonstration. The object of the paper which we here lay before the readers of the Journal, is to develop a fundamental idea which characterizes the processes of integration given by Monge and Lagrange, whatever be the order of the equation or the number of the variables, and to shew by à priori considerations that the subsidiary assumptions do not involve any sacrifice of generality whatever.

A. We shall commence with the discussion of partial differential equations of the first order between three variables x, y, z. The general form of this class of equations is

$$Pp + Qq = R \ldots\ldots\ldots\ldots (1),$$

where P, Q, R, are given functions of x, y, z, and p, q, represent $\frac{dz}{dx}, \frac{dz}{dy}$, the partial differential coefficients of z.

Since z is a function of x and y, we have

$$dz = pdx + qdy \ldots\ldots\ldots\ldots (2).$$

Eliminating q between (1) and (2) we get

$$(Pdy - Qdx)\,p = Rdy - Qdz \ldots\ldots (3).$$

Now since the two ratios subsisting between the three differentials dx, dy, dz, are subject to only a single equation (2), it is evident that we may establish any relation whatever between either of these ratios and the variables x, y, z, without restricting in any degree the absolute or relative values of these variables, and therefore without limiting the generality of the equation (1): assume then

$$Pdy - Qdx = 0 \ldots\ldots\ldots\ldots (4),$$

and therefore, by (3),

$$Rdy - Qdz = 0 \ldots\ldots\ldots\ldots (5).$$

Suppose now that from these two equations, (4) and (5), we can obtain two perfect differentials

$$dF = 0, \quad dG = 0,$$

where F, G, are certain functions of x, y, z. Since these two equations are simultaneous, or, which is the same thing, since $dF = 0$ whenever $dG = 0$, and vice versâ, it is evident that F and G must be functional of each other.

Hence we see that from an assumption involving no sacrifice of generality we obtain

$$\phi\,(F, G) = 0,$$

which is consequently a relation among the three variables x, y, z, coextensive in generality with the equation (1), and therefore the complete integral of the proposed equation. The symbol ϕ evidently denotes undefined and therefore arbitrary functionality.

The nature of the assumption on which this method of integration hinges may receive a convenient illustration from the algebraic geometry of surfaces. Let x, y, z, be the co-ordinates of any point in a surface, and let θ denote the angle between the axis of x and the projection upon the plane of x, y, of an elemental arc between this point and a contiguous point $x + dx$, $y + dy$, $z + dz$; then, the number of contiguous points being infinite, it is clear that we are at liberty to pass to a new point such that $\tan\theta \left(= \frac{dy}{dx}\right) = \frac{Q}{P}$: this is admissible at whatever point x, y, z, we choose to start, and does not presume any peculiarity in the form of the surface. For the success of Lagrange's integration, it is necessary that from the two auxiliary equations (4) and (5) it be possible to obtain an equation containing only two variables, the differentials of which it involves. Probably when this is not possible, the auxiliary equations may correspond to a discontinuous line traced on the surface.

B. The general form of a partial differential equation of the second order and first degree between three variables is

$$Pr + Qs + Rt = S \ldots\ldots\ldots\ldots (1),$$

where P, Q, R, S, designate assigned functions of x, y, z, p, q, and where r, s, t, denote respectively the partial differential coefficients

$$\frac{d^2z}{dx^2},\ \frac{d^2z}{dx\,dy},\ \frac{d^2z}{dy^2}.$$

By the principles of differentiation we have

$$dz = pdx + qdy \ldots\ldots\ldots\ldots (2),$$

$$dp = rdx + sdy \ldots\ldots\ldots\ldots (3),$$

$$dq = sdx + tdy \ldots\ldots\ldots\ldots (4).$$

Eliminating r and t between the three equations (1), (3), (4), we shall get

$$Pdy\,dp + Rdx\,dq - Sdx\,dy = s\,(Pdy^2 - Qdx\,dy + Rdx^2) \ldots (5).$$

But since the four ratios subsisting between the five differentials dx, dy, dz, dp, dq, are subject only to three equations (2), (3), (4), it is evident that we may establish any relation whatever between any one of these ratios and the variables x, y, z, p, q, without restricting in any degree the absolute or

relative values of x, y, z, p, q, r, s, t, and therefore without violating the generality of the proposed equation (1). Assume then

$$Pdy^2 - Qdx\,dy + Rdx^2 = 0 \;\ldots\ldots\ldots\; (6);$$

whence, by (5), we have

$$Pdy\,dp + Rdx\,dq = Sdx\,dy \;\ldots\ldots\; (7).$$

Now, from (6), by the solution of a quadratic, we shall have generally two values of $\frac{dy}{dx}$. Let k', k'', denote the two values, which will generally be functions of x, y, z, p, q. Then, from (6) and (7), we shall have the two following systems of simultaneous equations,

$$\left.\begin{aligned} dy &= k'dx, \\ Pk'dp + Rdq &= Sk'dx \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\; (8);$$

$$\left.\begin{aligned} dy &= k''dx, \\ Pk''dp + Rdq &= Sk''dx. \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\; (9).$$

Suppose that from (8) we can obtain two perfect differentials

$$dF = 0, \quad dG = 0,$$

where F, G, are certain functions of x, y, z, p, q. Then, by the reasoning employed in the integration of the equation of the first order, we shall have, as a first integral coextensive in generality with the proposed equation (1),

$$\phi\,(F, G) = 0 \;\ldots\ldots\ldots\ldots\; (10),$$

where the symbol ϕ denotes arbitrary functionality.

In the same way from the system (9) we may get

$$\phi_1\,(F_1, G_1) = 0 \;\ldots\ldots\ldots\ldots\; (11),$$

ϕ_1, F_1, G_1, corresponding to ϕ, F, G.

The integration of either of the equations (10) and (11) will give rise to an equation in x, y, z, which will be the same in both cases, as will be evident when it is considered that each of them is as general as the equation (1).

If it be more convenient, which is generally the case, we may combine the equations (10) and (11) as simultaneous, in order to obtain the complete integral. The admissibility of this arises from the fact that the complete integral must be such as to satisfy each of them singly, and therefore any equations resulting from their combination; the converse proposition being equally clear, viz. that any relation between x, y, z, which shall satisfy any two equations deduced from (10) and (11), shall likewise satisfy the equations (10) and (11) themselves.

We have not yet taken into consideration the difficulty started by Professor De Morgan regarding the possibility of

s becoming infinite when a is assumed equal to zero. Since however s is some function of x, y, z, p, q, and since the assumption $a = 0$, as we have shewn, does not in any way restrict the relations existing among these quantites, the difficulty ceases. It is not necessary to dwell on the cases in which the method of Monge fails, the ordinary treatises being sufficiently explicit on these points. We will proceed now to equations of the first order involving any number of variables.

C. The general form of a partial differential equation of the first order involving any number of variables $x_1, x_2, x_3, \ldots x_n, u$, is, u being reckoned the dependent variable,

$$P_1p_1 + P_2p_2 + P_3p_3 + \ldots . + P_np_n = Q \ldots . (1),$$

where $P_1, P_2, P_3, \ldots . P_n$, are proposed functions of $x_1, x_2, x_3, \ldots . x_n, u$, and $p_1, p_2, p_3, \ldots . p_n$, denote respectively the partial differential coefficients

$$\frac{du}{dx_1}, \frac{du}{dx_2}, \frac{du}{dx_3}, \ldots . \frac{du}{dx_n}.$$

Now we have

$$du = p_1dx_1 + p_2dx_2 + p_3dx_3 + \ldots . + p_n\,dx_n \ldots . (2);$$

hence, from (1), eliminating p_1,

$$P_1du - Qdx_1 + (P_2dx_1 - P_1dx_2)p_2 + (P_3dx_1 - P_1dx_3)p_3 + \ldots .$$
$$+ (P_ndx_1 - P_1dx_n)\,p_n = 0 \ldots \ldots (3).$$

Now since the equation (2) involves, beside the quantities $p_1, p_2, p_3, \ldots . p_n$, the n ratios subsisting between the $n + 1$ differentials $dx_1, dx_2, dx_3, \ldots . dx_n, du$, it is clear that we may establish arbitrarily between each of any $n - 1$ of the ratios and the variables $x_1, x_2, x_3, \ldots . x_n, u$, any relation we please without in any degree limiting the absolute or relative values of $x_1, x_2, x_3, \ldots . x_n, u, p_1, p_2, p_3, \ldots . p_n$, and therefore it is manifest that the assumption of these relations will in no wise restrict the generality of the proposed equation (1). Assume then

$$P_2dx_1 - P_1dx_2 = 0, \quad P_3dx_1 - P_1dx_3 = 0, \ldots . P_ndx_1 - P_1dx_n = 0,$$

and therefore from (3) we have also

$$P_1du - Qdx_1 = 0.$$

Suppose that from these n simultaneous equations we can obtain n perfect differentials

$$dF_1 = 0, \quad dF_2 = 0, \quad dF_3 = 0 \ldots . dF_n = 0.$$

These equations shew that our auxiliary assumptions, which we have shewn to introduce no restriction among the variables of the proposed equation (1), are equivalent to assuming any

one of the functions $F_1, F_2, F_3, \ldots F_n$, to be an undefined or arbitrary function of all the rest; hence the complete integral of (1) will be
$$\phi(F_1, F_2, F_3, \ldots F_n) = 0,$$
ϕ denoting arbitrary functionality.

D. The general form of a partial differential equation of the n^{th} order and of the first degree between three variables x, y, z, is

$$\overset{n}{\underset{0}{u}} \, . \, \overset{n}{\underset{0}{z}} + \overset{n-1}{\underset{1}{u}} \, . \, \overset{n-1}{\underset{1}{z}} + \overset{n-2}{\underset{2}{u}} \, . \, \overset{n-2}{\underset{2}{z}} + \ldots\ldots + \overset{1}{\underset{n-1}{u}} \, . \, \overset{1}{\underset{n-1}{z}} + \overset{0}{\underset{n}{u}} \, . \, \overset{0}{\underset{n}{z}} = v \ldots\ldots (1),$$

$\overset{n-r}{\underset{r}{z}}$ denoting $\dfrac{d^n z}{dx^{n-r} dy^r}$ and $\overset{n-r}{\underset{r}{u}}$ as well as v representing some function of x, y, z, and the partial differential coefficients of orders inferior to the n^{th}.

Then, corresponding to the equations $(A, 3)$ and $(A, 4)$, we shall have

$$d\overset{n-1}{\underset{0}{z}} = \overset{n}{\underset{0}{z}}\, dx + \overset{n-1}{\underset{1}{z}}\, dy,$$

$$d\overset{n-2}{\underset{1}{z}} = \overset{n-1}{\underset{1}{z}}\, dx + \overset{n-2}{\underset{2}{z}}\, dy,$$

$$d\overset{n-3}{\underset{2}{z}} = \overset{n-2}{\underset{2}{z}}\, dx + \overset{n-3}{\underset{3}{z}}\, dy,$$

$$\ldots\ldots\ldots$$

$$\ldots\ldots\ldots$$

$$d\overset{0}{\underset{n-1}{z}} = \overset{1}{\underset{n-1}{z}}\, dx + \overset{0}{\underset{n}{z}}\, dy;$$

hence from (1) we get

$$\overset{0}{\underset{n}{z}} \left\{ \overset{0}{\underset{n}{u}}\, dx^n - \overset{1}{\underset{n-1}{u}}\, dx^{n-1} dy + \overset{2}{\underset{n-2}{u}}\, dx^{n-2} dy^2 - \ldots + (-)^n \overset{n}{\underset{0}{u}}\, dy^n \right\} = w,$$

where w represents an expression involving no partial differential coefficient of z of an order higher than the $(n-1)^{\text{th}}$, being a function of these coefficients and of their total differentials as well as of x, y, z.

Then, corresponding to the relations (6) and (7) of (B), we shall have

$$\overset{0}{\underset{n}{u}} \, . \, dx^n - \overset{1}{\underset{n-1}{u}}\, dx^{n-1} dy + \overset{2}{\underset{n-2}{u}}\, dx^{n-2} dy^2 - \ldots\ldots + (-)^n \overset{n}{\underset{0}{u}}\, dy^n = 0 \ldots\ldots (2),$$

$$w = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (2), an equation of the n^{th} degree, we can obtain n values of $\dfrac{dy}{dx}$, and therefore, corresponding to the two systems

of simultaneous equations (B, 8) and (B, 9), we shall have n such systems; and the remarks which we made respecting the pair of systems and their integrals, are evidently applicable mutatis mutandis to the more general case of the n systems.

E. We have in the preceding pages discussed the solutions of partial differential equations of all orders for three variables, and of the first order for any number of variables. It is unnecessary to extend this paper to greater length by the consideration of the general case of an equation of any number of variables and of any order, the principles which have been developed above being evidently universally applicable.

VII.—ON THE ATTRACTIONS OF CONDUCTING AND NON-CONDUCTING ELECTRIFIED BODIES.

In measuring the action exerted upon an electrified body, by a quantity of free electricity distributed in any manner over another body, the methods followed in the cases in which the attracted body is conducting and non-conducting are different. Now, the only difference between the state of a conducting body and that of a non-conducting body is, that the electricity is held upon a conducting body by the pressure of the atmosphere (to a certain extent at least), while on a non-conducting body it is held by the *friction* of the particles of the body.

To find the attraction of an electrical mass E, on a non-conducting electrified body A, the obvious way is to proceed as in ordinary cases of attraction, considering the electricity on A as the attracted mass.

In finding the action on a conducting body A, the method followed is to consider its electricity as exerting no pressure upon the particles of the body, but to disturb its equilibrium, by making the pressure of the air unequal at different parts of its surface. These two methods of measuring the action of E on A *should* obviously lead to the same result, since the action must be the same, whether A be conducting or non-conducting, the distribution remaining the same. It is the object of the following paper to show that they *do* lead to the same result.

We must first find the pressure of an element of the electricity of A, on the atmosphere.

Let ds be the area of the element, and ρds its electrical mass. Let ds form part of another element σ, indefinitely larger than ds in every direction, but so small that it may be considered as plane. Now, if $\rho\sigma$ be a material plane, it can exercise no attraction on ρds, in a direction perpendicular to

the plane, and it may be readily shown that this is also true if $\rho\sigma$ be a plate of matter of different densities, arranged in parallel planes, the thickness being either finite or indefinitely small, and the law of density being any whatever.

Hence, the force acting on ρds is due to the repulsion of all the electrical mass, except σ; and, since the electricity on A is in equilibrium under the influence of E, the repulsion acts along the normal through ds, and is in magnitude $2\pi\rho^2 ds$, (see vol. III. p. 75), which is therefore the pressure of ds on the air. Hence, if p be the barometric pressure of the atmosphere, the pressure on ds, perpendicular to the surface, is

$$(p - 2\pi\rho^2)\, ds.$$

Hence, if X be the whole pressure on A, resolved along a fixed line $X'X$, and if ν be the angle which the normal through ds makes with this line, we have

$$X = -\iint (p - 2\pi\rho^2) \cos\nu\, ds,$$

the integrals being extended over the surface of A. Now,

$$\iint p \cos\nu\, ds = 0,$$

since the pressure of the atmosphere does not disturb the equilibrium of A. Hence, we have

$$X = 2\pi \iint \rho^2 \cos\nu\, ds \ldots\ldots\ldots\ldots\ldots (a),$$

which is the expression for the attraction on a conducting body A, either separate from the body on which E is distributed, or connected with it.

To show that this is identical with the expression for the attraction of E on the electricity of A, let $R\rho ds$ and $R'\rho ds$ be the components of the repulsion on ρds, which are due to E, and to the electricity of A; and let a, a' be the angles which their directions make with XX'. Then we shall have

$$2\pi\rho \cos \nu = R \cos a + R' \cos a';$$

therefore $\qquad X = \iint (R \cos a + R' \cos a')\, ds.$

Now, $\iint R' \cos a' ds$ is the attraction of the electricity of A on itself in the direction XX', and is therefore $= 0$. Hence,

$$X = \iint R \cos a ds \ldots\ldots\ldots\ldots\ldots (b).$$

But this expression for X is the attraction of E on the electricity of A, and hence the two methods of measuring the action lead to the same result.

P. Q. R.

VIII.—REMARKS ON A THEOREM OF M. CATALAN.

By George Boole.

M. Catalan, in *Liouville's Journal*, vol. vi. p. 81, pursuing the consequences of the celebrated theorem of Dirichlet, has arrived at the following result: *If V be the value of the definite multiple integral,*

$$\iint .. \frac{dx_1 dx_2 .. dx_{n-1}}{x_n} f(a_1 x_1 + a_2 x_2 .. + a_n x_n) \ldots (1);$$

where $x_n^2 = 1 - x_1^2 - x_2^2 .. - x_{n-1}^2$, and the integrations are extended to all positive values of the $n-1$ independent variables $x_1, x_2, .. x_{n-1}$, subject to the condition

$$x_1^2 + x_2^2 .. + x_{n-1}^2 \overset{=}{<} 1;$$

then
$$V = \frac{\pi^{\frac{n-1}{2}}}{2^{n-1}\Gamma\left(\frac{n-1}{2}\right)} \int_0^{\frac{1}{2}\pi} d\theta (\sin\theta)^{n-2} f(A\cos\theta) \ldots (2).$$

From this result, which I shall prove to be erroneous, the author incorrectly deduces, as a particular consequence, the theorem of Poisson,

$$\int_0^\pi \int_0^{2\pi} dudv \sin u f(a_1 \cos u + a_2 \sin u \cos v + a_3 \sin u \sin v)$$
$$= 2\pi \int_0^\pi d\theta \sin\theta f(A\cos\theta) \ldots\ldots (3);$$

well known as the foundation of his solution of the partial differential equation

$$\frac{d^2u}{dt^2} = a^2 \left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right),$$

which occupies so conspicuous a place in the mathematical theories of Heat, Attraction, and Electricity. On account of the great importance of this application, and of the interest attaching to the subject of definite integrals generally, I purpose here to shew in what respects M. Catalan's demonstration is unsound, and to investigate the true form of the theorem of which Poisson's is a particular case.

M. Catalan assumes a new set of n variables $u_1, u_2, .. u_n$, connected with $x_1, x_2, .. x_n$, by linear equations, of which one is

$$a_1 x_1 + a_2 x_2 .. + a_n x_n = A u_1,$$

where $A = \sqrt{(a_1^2 + a_2^2 .. + a_n^2)}$, and the rest are so determined as to satisfy the condition

$$x_1^2 + x_2^2 .. + x_n^2 = u_1^2 + u_2^2 .. + u_n^2.$$

He thus obtains for the transformed integral,

$$V = \iint .. \frac{du_1 du_2 .. du_{n-1}}{u_n} f(Au_1) ....(4),$$

which he assumes to extend to all positive values of $u_1, u_2 ... u_{n-1}$, satisfying the condition

$$u_1^2 + u_2^2 .. + u_{n-1}^2 \leqq 1 ....(5).$$

Now, neither of the particulars involved in this assumption is realized. The equation of the limits is not

$$u_1^2 + u_2^2 .. + u_{n-1}^{2\,2} \leqq 1 ;$$

neither do the positive values of x_1, x_2, .. x_{n-1}, invariably correspond to positive values of u_1, u_2, .. u_{n-1}. The erroneous character of M. Catalan's conclusions will be more distinctly seen, if we assume a particular form of f, and value of n, and calculate the two values of V above quoted. Let $n = 3$, and $f(t) = t$, then by the above transformation we should have

$$\iint \frac{dx_1 dx_2}{\sqrt{(1 - x_1^2 - x_2^2)}} \{a_1 x_1 + a_2 x_2 + a_3 \sqrt{(1 - x_1^2 - x_2^2)}\}$$

$$= \iint \frac{du_1 du_2}{\sqrt{(1 - u_1^2 - u_2^2)}} \{A\sqrt{(1 - u_1^2 - u_2^2)}\} = A \iint du_1 du_2 ....(6);$$

the integrations extending to all positive values of the variables satisfying the conditions $x_1^2 + x_2^2 \leqq 1$, $u_1^2 + u_2^2 \leqq 1$. Now, by Liouville's extension of Dirichlet's theorem, if U be the value of the definite integral,

$$\iint .. \, dx_1 dx_2 .. x^{a-1} y^{b-1} .. f\left\{\left(\frac{x}{\alpha}\right)^p + \left(\frac{y}{\beta}\right)^q + \&c.\right\},$$

taken through the positive limits of the inequality,

$$\left(\frac{x}{\alpha}\right)^p + \left(\frac{y}{\beta}\right)^q + \&c. \leqq 1;$$

then $$U = \frac{\alpha^a \beta^b ..}{pq ..} \frac{\Gamma\left(\frac{a}{p}\right) \Gamma\left(\frac{b}{q}\right) ..}{\Gamma\left(\frac{a}{p} + \frac{b}{q} + ..\right)} \int_0^1 du f(u) u^{\frac{a}{p} + \frac{b}{q} .. - 1} ....(7).$$

Applying this theorem, we get for the first member of (6)

$$V = \frac{a_1 + a_2}{4} \frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \int_0^1 du \, u^{\frac{1}{2}} (1 - u)^{-\frac{1}{2}} + \frac{a_3 \pi}{4}$$

$$= \frac{a_1 + a_2 + a_3}{4} \pi,$$

since $\Gamma(\tfrac{1}{2}) = \pi^{\frac{1}{2}}$, and $\int_0^1 du\, u^{\frac{1}{2}} (1-u)^{-\frac{1}{2}} = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{2})}{\Gamma(2)} = \frac{\pi}{2}$.
But from the second member of (6), as from the second member of (2), we have

$$V = \frac{\sqrt{(a_1^2 + a_2^2 + a_3^2)}}{4}\,\pi;$$

and these results cannot be made to correspond, except for particular values of a_1, a_2, a_3, which are supposed to be entirely arbitrary.

I shall now develop what I conceive to be the proper mode of investigation.

Let $f'(t) = \frac{df(t)}{dt}$, and let us consider the definite multiple integral

$$\iint \ldots dx_1 dx_2 \ldots dx_n f'(a_1x_1 + a_2x_2 \ldots a_nx_n) = V \ldots (8),$$

the integrations extending to all real values, positive or negative, of the n independent variables, satisfying the condition

$$x_1^2 + x_2^2 \,.\,. + x_n^2 \overline{\overline{<}}\, 1 \ldots\ldots (9).$$

Let the system $x_1, x_2 \ldots x_n$, be linearly transformed into the system u_1, $u_2 \ldots u_n$, in accordance with the conditions

$$\left.\begin{aligned} a_1x_1 + a_2x_2 \ldots + a_nx_n &= Au_n, \\ x_1^2 + x_2^2 \ \ \ldots + \ x_n^2 &= u_1^2 + u_2^2 \ldots + u_n^2, \end{aligned}\right\} \ldots . (10).$$

We shall obviously have $A = \sqrt{(a_1^2 + a_2^2 \ldots . + a_n^2)}$, and $dx_1dx_2 \ldots dx_n = du_1du_2 \ldots du_n$; whence

$$V = \iint \ldots du_1du_2 \ldots du_n f'(Au_n) \ldots . (11),$$

the integrations extending to all real values of u_1, $u_2 \ldots . u_n$, subject to the condition

$$u_1^2 + u_2^2 \ldots + u_n^2 \overline{\overline{<}}\, 1 \ldots\ldots\ldots (12).$$

Integrating with respect to u_n between the limits

$$u_n = -\sqrt{(1 - u_1^2 \ldots - u_{n-1}^2)} \text{ and } u_n = \sqrt{(1 - u_1^2 \ldots - u_{n-1}^2)},$$

we have

$$V = \iint \ldots du_1du_2 \ldots du_{n-1} \frac{\{f(A\sqrt{1-u_1^2 \ldots - u_{n-1}^2}) - f - (A\sqrt{1-u_1^2 \ldots - u_{n-1}^2})\}}{A} \ldots\ldots (13);$$

the equation of the limits now becoming

$$u_1^2 + u_2^2 \,.\,. + u_{n-1}^2 \overline{\overline{<}}\, 1 \ldots\ldots\ldots (14).$$

Now, by Liouville's theorem (7)

$$\iint \ldots du_1\, du_2 \ldots du_{n-1} f\{A\sqrt{(1 - u_1^2 \,.\,. - u_{n-1}^2)}\}$$

$$= \frac{1}{2^{n-1}} \frac{\{\Gamma(\frac{1}{2})\}^{n-1}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^1 dv\, v^{\frac{n-3}{2}} f\{A\sqrt{(1-v)}\},$$

$$\iint \ldots du_1\, du_2 \ldots du_{n-1} f\{-A\sqrt{(1-u_1^2 \ldots - u_{n-1}^2)}\}$$
$$= \frac{1}{2^{n-1}} \frac{\{\Gamma(\frac{1}{2})\}^{n-1}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^1 dv\, v^{\frac{n-3}{2}} f\{-A\sqrt{(1-v)}\};$$

the integrations being extended through the positive values of $u_1, u_2 \ldots u_{n-1}$ only. But $u_1, u_2 \ldots u_{n-1}$ enter under the sign of integration in even powers. Hence the value of V will be found by substituting the above expressions in the second member of (13), and multiplying the result by 2^{n-1}, which will extend the integrations to all real values of $u_1, u_2 \ldots u_{n-1}$. We thus find

$$V = \frac{\{\Gamma(\frac{1}{2})\}^{n-1}}{A\Gamma\left(\frac{n-1}{2}\right)} \int_0^1 dv\, v^{\frac{n-3}{2}} [f\{A\sqrt{(1-v)}\} - f\{-A\sqrt{(1-v)}\}] \ldots (14).$$

Let $v = (\sin\theta)^2$, then $dv = 2\sin\theta\cos\theta d\theta$, $\sqrt{(1-v)} = \cos\theta$, also $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$, therefore

$$V = \frac{2\pi^{\frac{n-1}{2}}}{A\Gamma\left(\frac{n-1}{2}\right)} \left\{ \int_0^{\frac{1}{2}\pi} d\theta\, (\sin\theta)^{n-2} \cos\theta\, f(A\cos\theta) - \int_0^{\frac{1}{2}\pi} d\theta\, (\sin\theta)^{n-2} \cos\theta\, f(-A\cos\theta) \right\}.$$

In the second integral,
let $\theta = \pi - \theta'$, then $d\theta = -d\theta'$, $\cos\theta = -\cos\theta'$, $\sin\theta = \sin\theta'$, the limits of θ' being π and $\frac{1}{2}\pi$, hence

$$-\int_0^{\frac{1}{2}\pi} d\theta (\sin\theta)^{n-2} \cos\theta f(-A\cos\theta) = -\int_\pi^{\frac{1}{2}\pi} d\theta' \sin\theta'^{n-2} \cos\theta' f(A\cos\theta')$$
$$= \int_{\frac{1}{2}\pi}^{\pi} d\theta' \sin\theta'^{n-2} \cos\theta' f(A'\cos\theta').$$

Taking away the accent from θ', and adding to the first integral in the value of V, we find

$$V = \frac{2\pi^{\frac{n-1}{2}}}{A\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi d\theta\, (\sin\theta)^{n-2} \cos\theta\, f(A\cos\theta) \ldots\ldots (15).$$

Now resuming (8), integrate with respect to x_n between the limits $x_n = -\sqrt{(1-x_1^2 \ldots - x_{n-1}^2)}$ and $x_n = \sqrt{(1-x_1^2 \ldots - x_{n-1}^2)}$, we have

$$V = \iint \ldots \frac{dx_1 dx_2 \ldots dx_{n-1}}{a_n} \left\{ \begin{array}{l} f\{a_1 x \ldots + a_{n-1}x_{n-1} + a_n\sqrt{(1-x_1^2 \ldots - x_{n-1}^2)}\} \\ - f\{a_1 x_1 \ldots + a_{n-1}x_{n-1} - a_n\sqrt{(1-x_1^2 \ldots - x_{n-1}^2)}\} \end{array} \right\}$$
$$\ldots\ldots (16),$$

the integrations now extending to all real values of $x_1, x_2 .. x_{n-1}$, satisfying the condition

$$x_1^2 + x_2^2 .. + x_{n-1}^2 \overline{\overline{<}} 1.$$

Comparing this result with (15), and writing V in the place of $a_n V$, we have the following general theorem.

THEOREM. *If V be the value of the definite multiple integral,*

$$\int\int .. dx_1 dx_2 .. dx_{n-1} \{f(a_1 x_1 .. + a_{n-1} x_{n-1} + a_n x_n) \\ - f(a_1 x_1 .. + a_{n-1} x_{n-1} - a_n x_n)\},$$

extending to all real values of $x_1, x_2 .. x_{n-1}$ *and to all real and positive values of* x_n, *which satisfy the conditions*

$$x_1^2 + x_2^2 .. + x_{n-1}^2 \overline{\overline{<}} 1,$$

$$x_1^2 + x_2^2 .. + x_n^2 = 1,$$

then $$V = \frac{2a_n \pi^{\frac{n-1}{2}}}{A\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi d\theta\, (\sin\theta)^{n-3} \cos\theta\, f(A\cos\theta) \;..\; (17).$$

In the case of $n = 3$, I shall now shew that this may be resolved into a beautiful system of four allied theorems, of which that of Poisson is one.

We have first to consider the expression

$$\int\int dx_1 dx_2 \{f(a_1 x_1 + a_2 x_2 + a_3 x_3) - f(a_1 x_1 + a_2 x_2 - a_3 x_3)\} \;..\; (18).$$

Let $x_1 = \cos u$, $x_2 = \sin u \cos v$, $x_3 = \sin u \sin v$. Transforming in the usual way, $dx_1 dx_2 = (\sin u)^2 \sin v\, du\, dv$. To find the limits we may reason thus: The variables x_1, x_2, must each, independently of the sign of the other, admit of all values from -1 to 1, and the dependent variable x_3, of all values from 0 to 1. This can only be effected by assigning to both u and v the limits 0 and π; whence

$$V = \int_0^\pi \int_0^\pi du\, dv\, (\sin u)^2 \sin v\, f(a_1 \cos u + a_2 \sin u \cos v + a_3 \sin u \sin v)$$

$$- \int_0^\pi \int_0^\pi du\, dv\, (\sin u)^2 \sin v\, f(a_1 \cos u + a_2 \sin u \cos v - a_3 \sin u \sin v).$$

In the second integral put $v = 2\pi - v'$, and transforming, we get

$$- \int_0^\pi \int_{2\pi}^\pi du\, dv' (\sin u)^2 \sin v' f(a_1 \cos u + a_2 \sin u \cos v' + a_3 \sin u \sin v')$$

$$= \int_0^\pi \int_\pi^{2\pi} du\, dv' (\sin u)^2 \sin v' f(a_1 \cos u + a_2 \sin u \cos v' + a_3 \sin u \sin v')$$

which, on omitting the accent, and connecting the result with the first integral in the value of V, will give

$$V=\int_0^{\pi}\int_0^{2\pi} du\,dv(\sin u)^2\sin v f(a_1\cos u+a_2\sin u\cos v+a_3\sin u\sin v).$$

Let $a_1\cos u + a_2\sin u\cos v + a_3\sin u\sin v = U$, then comparing the last result with (17),

$$\int_0^{\pi}\int_0^{2\pi} du dv(\sin u)^2\sin v f(U)=\frac{2a_3\pi}{A}\int_0^{\pi} d\theta\sin\theta\cos\theta f(A\cos\theta)..(\mathrm{I}),$$

which is the first of the four theorems above alluded to.

If we now take $x_2\, x_3$ as the independent variables, subject to the condition, $x_2^2+x_3^2 \overline{\lessdot} 1$, and determine x_1 by the equation $x_1^2+x_2^2+x_3^2=1$; we shall obviously have, by inspection of (17),

$$\iint dx_2\, dx_3\,\{f(a_2x_2+a_3x_3+a_1x_1)-f(a_2x_2+a_3x_3-a_1x_1)\}$$
$$=\frac{2a_1\pi}{A}\int_0^{\pi} d\theta\sin\theta\cos\theta f(A\cos\theta).$$

From the values of x_2, x_3, above assumed, we have

$$dx_2\, dx_3=\sin u\cos u\, du\, dv.$$

Here x_1 is to be positive, while x_2, x_3 independently admit all values from -1 to $+1$. These conditions give for the limits of u, 0 and $\frac{1}{2}\pi$, for those of v, 0 and 2π. Hence

$$V=\int_0^{\frac{1}{2}\pi}\int_0^{2\pi} du\,dv\sin u\cos u f(a_2\sin u\cos v+a_3\sin u\sin v+a_1\cos u),$$
$$-\int_0^{\frac{1}{2}\pi}\int_0^{2\pi} du\,dv\sin u\cos u f(a_2\sin u\cos v+a_3\sin u\sin v-a_1\cos u).$$

In the second integral let $u=\pi-u'$; and proceeding as before, we finally get

$$\int_0^{\pi}\int_0^{2\pi} du\,dv\sin u\cos u f(U)=\frac{2a_1\pi}{A}\int_0^{\pi} d\theta\sin\theta\cos\theta f(A\cos\theta) \quad \ldots.(\mathrm{II}),$$

which is the second theorem of the system.

Next, let x_3, x_1 be the independent variables, x_2 being given by the condition $x_1^2+x_2^2+x_3^2=1$. Proceeding as in the two last cases, we have for our third theorem

$$\int_0^{\pi}\int_0^{2\pi} du\,dv(\sin u)^2\cos v f(U)=\frac{2a_2\pi}{A}\int_0^{\pi} d\theta\sin\theta\cos\theta f(A\cos\theta) \quad \ldots.(\mathrm{III}).$$

Now multiply (II) by a_1, (III) by a_2, (I) by a_3, and add the results, we have

$$\int_0^{\pi}\int_0^{2\pi} du dv\sin u\,(a_1\cos u+a_2\sin u\cos v+a_3\sin u\sin v) f(U)$$
$$=\frac{2(a_1^2+a_2^2+a_3^2)\pi}{A}\int_0^{\pi} d\theta\sin\theta\cos\theta f(A\cos\theta);$$

or $\int_0^\pi\int_0^{2\pi} dudv \sin u\, Uf(U) = 2A\pi\int_0^\pi d\theta \sin\theta\cos\theta f(A\cos\theta)$.

Now the function f being arbitrary, for $Uf(U)$ write $f(U)$, and for $A\cos\theta f(A\cos\theta)$ write $f(A\cos\theta)$, then

$$\int_0^\pi\int_0^{2\pi} dudv \sin u f(U) = 2\pi\int_0^\pi d\theta \sin\theta f(A\cos\theta)\ldots\ldots(\text{IV});$$

which is Poisson's theorem, the last of the allied system.

It is worthy of observation that I, II, III, may be deduced from IV, by differentiating with respect to the constants.

The following result, of which and of some others the demonstration is reserved for a future occasion, constitutes the generalization of IV, and is deduced from the same general formula of definite integration (17). viz

$$\int_0^\pi\int_0^\pi\ldots\int_0^{2\pi} d\theta_1 d\theta_2\ldots d\theta_n U_n f\{\Sigma_{r=1}^{r=n+1}(a_r V_r)\}$$

$$= \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\int_0^\pi d\theta(\sin\theta)^{n-1} f(A\cos\theta),$$

wherein $U_n = (\sin\theta_1)^{n-1}(\sin\theta_2)^{n-2}\ldots(\sin\theta_{n-1})$,

and $V_r = \sin\theta_1\sin\theta_2\ldots\sin\theta_{r-1}\cos\theta_r$, up to V_n inclusive,

then $V_{n+1} = \sin\theta_1\sin\theta_2\ldots\sin\theta_n$;

moreover $A = (a_1^2 + a_2^2\ldots + a_{n+1}^2)^{\frac{1}{2}}$.

Lincoln, January 1843.

IX.—ON LAGRANGE'S THEOREM.

By ARTHUR CAYLEY, B.A. Fellow of Trinity College.

1. THE value given by Lagrange's theorem for the expansion of any function of the quantity (x), determined by the equation

$$x = u + hfx\ldots\ldots\ldots\ldots(1),$$

admits of being expressed in rather a remarkable symbolical form. The *à priori* deduction of this, independently of any expansion, presents some difficulties; I shall therefore content myself with showing that the form in question satisfies the equations

$$\frac{d}{du}\cdot\int F'x\,fx\,dx = \frac{d}{dh}\cdot\int F'x\,dx\ldots.(2),$$

$$Fx = Fu \text{ for } h = 0\ldots\ldots(3),$$

deduced from the equation (1). and which are sufficient to

determine the expansion of Fx, considered as a function of u and h in powers of h.

Consider generally the symbolical expression

$$\phi\left(h\frac{h}{dh}\right)\Xi(h)\ldots\ldots\ldots\ldots(4),$$

$\phi\left(h\dfrac{d}{dh}\right)$ involving in general symbols of operation relative to any of the other variables entering into $\Xi(h)$. Then, if $\Xi(h)$ be expansible in the form

$$\Xi(h)=\Sigma(Ah^m)\ldots\ldots\ldots\ldots(5),$$

it is obvious that

$$\phi.\left(h\frac{d}{dh}\right)\Xi(h)=\Sigma\{\phi m.(Ah^m)\}=\Sigma\{(\phi m.A)h^m\}..(6).$$

For instance, (u) representing a variable contained in the function $\Xi(h)$, and taking a particular form of $\phi\left(h\dfrac{d}{dh}\right)$,

$$\left(\frac{d}{du}\right)^{h\frac{d}{dh}}.\Xi(h)=\Sigma\left(\frac{d^m.A}{du^m}.h^m\right)\ldots\ldots\ldots(7).$$

From which it is easy to demonstrate

$$\frac{d}{du}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}.\Xi(h)\right\}=\frac{1}{h}\left(\frac{d}{du}\right)^{h\frac{d}{dh}}.\{h\Xi(h)\}....(8),$$

$$\frac{d}{dh}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}.\Xi(h)\right\}=\frac{1}{h}\left(\frac{d}{du}\right)^{h\frac{d}{dh}}.\{h\Xi'(h)\}....(9),$$

where $\Xi'h$ denotes $\dfrac{d}{dh}\Xi(h)$, as usual. Hence also

$$\frac{d}{du}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}.\Xi'h\right\}=\frac{d}{dh}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}.\Xi h\right\}....(10),$$

of which a particular case is

$$\frac{d}{du}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}.F'ufu\,e^{hfu}\right\}=\frac{d}{dh}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}.F'u.e^{hfu}\right\}..(11).$$

Also, $$\left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}(F'u\,e^{hfu})=Fu\quad\text{for } h=0..(12).$$

Hence the form in question for Fx is

$$Fx=\left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}(F'u\,e^{hfu})\ldots\ldots(13);$$

from which, differentiating with respect to u, and writing F instead of F',

$$\frac{Fx}{1 - hf'x} = \left(\frac{d}{du}\right)^{h\frac{d}{dh}} (Fu\, e^{hfu}) \ldots\ldots (14),$$

a well-known form of Lagrange's theorem, almost equally important with the more usual one. It is easy to deduce (13) from (14). To do this, we have only to form the equation

$$\frac{-hFx \,.\, f'x}{1 - hf'x} = -h\left(\frac{d}{du}\right)^{h\frac{d}{dh}} (Fuf'u\, e^{hfu}) \ldots (15),$$

deduced from (14) by writing $Fxf'x$ for fx, and adding this to (14),

$$\begin{aligned} Fx &= \left(\frac{d}{du}\right)^{h\frac{d}{dh}} (Fu\, e^{hfu}) - h\left(\frac{d}{du}\right)^{h\frac{d}{dh}} (Fuf'u\, e^{hfu}) \\ &= \left(\frac{d}{du}\right)^{h\frac{d}{dh}-1} \left\{\frac{d}{du}(Fu \,.\, e^{hfu}) - h \,.\, f'u\, Fu \,.\, e^{hfu}\right\} \\ &= \left(\frac{d}{du}\right)^{h\frac{d}{dh}-1} (F'u\, e^{hfu}) \ldots\ldots\ldots\ldots\ldots\ldots (16). \end{aligned}$$

In the case of several variables, if

$$\left.\begin{aligned} x &= u + hf(x,\ x_1 \ldots) \\ x_1 &= u_1 + h_1 f_1(x, x_1 \ldots) \end{aligned}\right\} \ldots\ldots (17),$$

writing for shortness

$$F, f, f_1 \ldots \text{ for } F(u, u_1 \ldots), f(u, u_1 \ldots), f_1(u, u_1 \ldots),$$

$$\frac{F(x, x_1 \ldots)}{\{1 - hf'(x)\}\{1 - h_1 f_1'(x_1) \ldots\}} = \left(\frac{d}{du}\right)^{h\frac{d}{dh}} \left(\frac{d}{du_1}\right)^{h_1\frac{d}{dh_1}} \ldots (F e^{hf + h_1 f_1 + \ldots}) \ldots (18),$$

$$\left\{\text{where } f'(x) = \frac{d}{dx} \,.\, f(x, x_1 \ldots), \text{ \&c.}\right\}$$

or the coefficient of $h^n \,.\, h_1^{n_1} \ldots$ in the expansion of

$$\frac{F(x, x_1 \ldots)}{\{1 - hf'(x)\}\ \{1 - h_1 f_1'(x_1)\} \ldots} \ldots\ldots\ldots (19)$$

is $$\frac{1}{1.2 \ldots n \,.\, 1.2 \ldots n_1} \left(\frac{d}{du}\right)^n \left(\frac{d}{du_1}\right)^{n_1} \ldots Ff^n \,.\, f_1^{n_1} \ldots\ldots (20).$$

From the formula (18), a formula may be deduced for the expansion of $F(x, x_1 \ldots)$, in the same way as (13) was deduced from (14), but the result is not expressible in a simple form by this method. An apparently simple form has indeed been given for this expansion by Laplace, *Mécanique Celeste*, tom. I. p. 176; but the expression there given for the general

term, requires first that certain differentiations should be performed, and then that certain changes should be made in the result, quantities $z, z' \ldots.$, which are to be changed into $z^n, z_{,}^{n\prime} \ldots.$; in other words, the general term is not really expressed by known symbols of operation only. The formula (18) is probably known, but I have not met with it anywhere.

X.—NOTE ON ORTHOGONAL ISOTHERMAL SURFACES.

THE object of this article is to state an important question relative to Orthogonal Isothermal Surfaces, which does not seem to have been yet considered generally.

It is known that if a system of surfaces be given, there are two, and only two, other systems cutting one another and the first system at right angles. Lamé has proved many general properties of such conjugate systems of surfaces, and has also considered particularly the case in which each of the three systems is a series of isothermal surfaces. He has nowhere however proved the possibility of the existence of three such systems of surfaces in general, or, in other words, he has not shown that if one system be isothermal, the two others which are determined by it are isothermal also. There is however a considerable probability that this proposition is generally true. For, in the first place, it is true, as will be shown below, of any system whatever of isothermal *cylindrical* surfaces, and it is also true of all isothermal surfaces of the second order, whether cylindrical or not. As these two cases, the only ones which have as yet been discussed, are very distinct in their nature, they afford a considerable presumptive evidence of the truth of the general proposition.

The case of isothermal surfaces of the second order has been fully discussed by Lamé, in a paper on Isothermal Surfaces in Liouville's Journal, (vol. II. p. 147), where he has shown that all *confocal* surfaces of the second order are isothermal. Now the two systems of orthogonal surfaces conjugate to a given series of confocal surfaces of the second order are also confocal surfaces of the second order, and therefore they are likewise isothermal. Hence the proposition is true for isothermal surfaces of the second order.

It may be proved in the following manner to be true for a series of isothermal cylindrical surfaces.

Of the two systems of orthogonal surfaces, conjugate to the given series of cylindrical surfaces, one is a series of cylindrical surfaces cutting them at right angles, and having their generating lines parallel to those of the first system, and the

other is a series of planes perpendicular to the cylinders. Now a series of parallel planes is obviously a system of isothermal surfaces, and it therefore only remains to be proved that, if one series of cylindrical surfaces be isothermal, the series which cuts them at right angles will be isothermal also.

To prove this, let us suppose any two surfaces of the first system to be retained at uniform temperatures, and let v be the temperature of any point (xy) between them. The equation of any one of the isothermal surfaces will be

$$v = a \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which is therefore the general equation, comprehending all surfaces of the first series.

By the equation of equilibrium of heat, moving in two directions, we must have

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0 \ldots\ldots\ldots\ldots\ldots (2).$$

Now let $$v_1 = a_1 \ldots\ldots\ldots\ldots\ldots (3)$$

be the general equation of the system of cylindrical surfaces, which cuts the first system at right angles.

Hence $$\frac{dv}{dx}\frac{dv_1}{dx} + \frac{dv}{dy}\frac{dv_1}{dy} = 0 \ldots\ldots\ldots\ldots (4),$$

Let $$\frac{dv_1}{dx} = k\frac{dv}{dy} \ldots\ldots\ldots (a),$$

therefore $$\frac{dv_1}{dy} = -k\frac{dv}{dx} \ldots\ldots (b).$$

In these equations k is arbitrary, with the exception that it must be so chosen that $\frac{dv_1}{dx}dx + \frac{dv_1}{dy}dy$ shall be a complete differential. Hence we must have

$$\frac{d}{dy}(a) - \frac{d}{dx}(b) = 0,$$

and therefore $$k\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}\right) + \frac{dk}{dx}\frac{dv}{dx} + \frac{dk}{dy}\frac{dv}{dy} = 0,$$

or, on account of (2),

$$\frac{dk}{dx}\frac{dv}{dx} + \frac{dk}{dy}\frac{dv}{dy} = 0.$$

This equation is satisfied if $k = 1$, and therefore we may use this value for k in (a) and (b). Hence

$$\frac{dv_1}{dx} = \frac{dv}{dy} \dots\dots\dots\dots\dots (c),$$

$$\frac{dv_1}{dy} = -\frac{dv}{dx} \dots\dots\dots\dots\dots (d).$$

These equations determine a form of v_1 which satisfies equation (4), the only condition to which v_1 is subject, in order that $v_1 = a_1$ may be the equation to the series of cylinders cutting the first series at right angles.

Now $$\frac{d}{dx}(c) + \frac{d}{dy}(d) \text{ gives}$$

$$\frac{d^2v_1}{dx^2} + \frac{d^2v_1}{dy^2} = 0 \quad \dots\dots\dots\dots (5).$$

Hence a function v_1 of x and y, which satisfies (5), may be found such that $v_1 = a_1$ represents the series of cylindrical surfaces cutting the first series at right angles. Hence the second system is isothermal.

The theorem which has just been proved, was first given by Lamé in a digression on Orthogonal Surfaces, contained in a paper entitled "*Mémoire sur les Lois de l'Equilibre du Fluide Ethéré*," in the *Journal de l'Ecole Polytechnique*, (vol. III. cahier XXIII). His proof however is deduced from some general properties of orthogonal surfaces which he has been discussing, and in a subsequent paper on the motion of heat in cylindrical bodies, (Liouville's Journal, vol. I.) he has not noticed the proposition at all.

P. Q. R.

XI.—ON FOURIER'S THEOREM.

The following demonstration of Fourier's Theorem for the transformation of a function may be interesting, as connecting it more directly with the Calculus of Operations and the ordinary series for the development of functions. It is not offered as eluding or overcoming the difficulties which seem to attend every demonstration of this important theorem, but it at least shows distinctly the nature of the difficulty to be overcome.

Let us consider the integral

$$\int da\, f(a) \cos q\,(x - a) = \int da\, f(a)\, \varepsilon^{-a\frac{d}{dx}} \cos qx,$$

by Taylor's theorem. But by the property of a differential of the product of two functions

$$\int da\, f(a)\, \epsilon^{-a\frac{d}{dx}} = \left(\frac{d}{da} + \frac{d'}{da}\right)^{-1} f(a)\, \epsilon^{-a\frac{d}{dx}},$$

where the accented letter refers to $f(a)$, and the unaccented to $\epsilon^{-a\frac{d}{dx}}$. Expanding the second side by the theorem of Bernoulli, we have

$$\int da f(a)\epsilon^{-a\frac{d}{dx}} = \left\{\left(\frac{d}{da}\right)^{-1} - \left(\frac{d}{da}\right)^{-2}\frac{d'}{da} + \left(\frac{d}{da}\right)^{-3}\left(\frac{d'}{da}\right)^{2} - \&c.\right\} f(a)\epsilon^{-a\frac{d}{dx}}.$$

And since

$$\left(\frac{d}{da}\right)^{-1} \epsilon^{-a\frac{d}{dx}} = \int da\, \epsilon^{-a\frac{d}{dx}} = -\left(\frac{d}{dx}\right)^{-1} \epsilon^{-a\frac{d}{dx}}, \text{ we have}$$

$$\int da f(a)\epsilon^{-a\frac{d}{dx}} = -\left(\frac{d}{dx}\right)^{-1}\left\{1 + \left(\frac{d}{dx}\right)^{-1}\frac{d}{da} + \left(\frac{d}{dx}\right)^{-2}\left(\frac{d}{da}\right)^{2} + \&c.\right\} f(a)\epsilon^{-a\frac{d}{dx}}.$$

Applying the operations on both sides to $\cos qx$, we find

$$\int da f(a)\cos q(x-a) = -\left(\frac{d}{dx}\right)^{-1}\left\{1 + \left(\frac{d}{dx}\right)^{-1}\frac{d}{da} + \&c.\right\} f(a)\cos q(x-a).$$

On taking this between the limits of $a = 0$ and $a = \infty$, and assuming that $\cos\infty$ is a zero of an order sufficient to destroy $f(\infty)$, this gives us

$$\int_0^\infty da\, f(a)\cos q(x-a) = \left(\frac{d}{dx}\right)^{-1}\left\{f(0) + \left(\frac{d}{dx}\right)^{-1} f'(0) + \&c.\right\}\cos qx,$$

$$\text{where}\quad f^{(n)}(0) = \left(\frac{d}{da}\right)^{n} f(a) \text{ when } a = 0.$$

Now if $u = \int_0^\infty dq \int_0^\infty da\, f(a)\cos q(x-a)$, we have

$$u = \left(\frac{d}{dx}\right)^{-1}\left\{f(0) + \left(\frac{d}{dx}\right)^{-1} f'(0) + \&c.\right\}\int_0^\infty dq\cos qx.$$

But $\left(\frac{d}{dx}\right)^{-1}\int_0^\infty dq\cos qx = \int_0^\infty \frac{dq}{q}\sin qx = \frac{\pi}{2}$,

and therefore

$$\left(\frac{d}{dx}\right)^{-(r+1)}\int_0^\infty dq\cos qx = \frac{\frac{1}{2}\pi x^r}{1.2\ldots.r},$$

$$\text{so that}\quad u = \frac{\pi}{2}\left\{f(0) + \frac{x}{1}f'(0) + \frac{x^2}{1.2}f''(0) + \&c.\right\}$$

The part within brackets is, by Maclaurin's theorem, equal to $f(x)$, and so

$$\int_0^\infty dq \int_0^\infty d\alpha\, f(\alpha) \cos q (x - \alpha) = \frac{\pi}{2} f(x),$$

which is Fourier's theorem.

It is to be observed, that the proof depends essentially on the assumption that the expression

$$f(\alpha) \cos q (x - \alpha)$$

vanishes when $\alpha = \infty$, which implies that $\cos \infty$ is a zero of a higher order than the infinity which may be the value of $f(\infty)$. This agrees with a remark made by Professor De Morgan, that Defler's verification depends on the assumption that $\sin \infty$ is a zero of the same order as $\varepsilon^{-\infty}$. (See his *Differential Calculus*, p. 628, note.)

G.

XII.—MATHEMATICAL NOTES.

1. *Stability of Eccentricities and Inclinations.* The equation for proving the stability of the eccentricites and inclinations of the planetary orbits may, as has been shown by Laplace, be deduced from the principle of the conservation of areas, joined to the fact of the invariability of the major axes.

Let m be the mass of a planet, h be twice the area described by the radius vector about the sun in an unit of time projected on the ecliptic, and i the inclination of its orbit to the ecliptic; then, by the principle of the conservation of areas,

$$\Sigma (mh) = \text{const.}$$

Every term under the sign of Summation is positive, because all the planets move round the sun in the same direction. But

$$h = \{a(1 - e^2)\}^{\frac{1}{2}} \cos i = a^{\frac{1}{2}} (1 - e^2)^{\frac{1}{2}} (1 + \tan^2 i)^{-\frac{1}{2}}.$$

Now e and i are small at the present time; hence, if we neglect their fourth power and the products of their squares, we have

$$h = a^{\frac{1}{2}} \{1 - \tfrac{1}{2} e^2 - \tfrac{1}{2} \tan^2 i\}.$$

Hence $$\Sigma \{ma^{\frac{1}{2}} (1 - \tfrac{1}{2} e^2 - \tfrac{1}{2} \tan^2 i)\} = \text{const.}$$

But since the major axes have no secular inequalities

$$\Sigma (ma^{\frac{1}{2}}) = \text{const.};$$

hence the preceding equation is equivalent to

$$\Sigma (ma^{\frac{1}{2}} e^2 + ma^{\frac{1}{2}} \tan^2 i) = \text{const.}$$

Now the left-hand side of the equation being small at the present time, the second side is also small, and therefore the first side is always small, and therefore

$$\Sigma\,(ma^{\frac{1}{2}}\,e^2) \text{ and } \Sigma\,(ma^{\frac{1}{2}}\tan^2 i)$$

are both always small. ε.

2. *Mnemonic Rule.*—The following mnemonic rule for the "Cotangent" formula in spherical trigonometry may be found useful.

If in any spherical triangle four parts be taken in succession, as for example, A, c, B, a, consisting of two means c, B, and two extremes A, a; then "The product of the cosines of the two means is equal to the sine of the mean *side* × cotangent of the extreme *side* − sine of the mean *angle* × cotangent of extreme *angle*." That is,

$$\cos c \cos B = \sin c \cot a - \sin B \cot A.$$

The negative sign may be got rid of by taking as the parts the two angles and the *supplements* of the two sides.

π.

3. To shew that the greatest and least radii of any plane section of the surface of elasticity are at right angles.

We have (*Math. Jour.* vol. I. p. 8.)

$$x = \frac{Al}{r^2 - a^2}, \quad y = \frac{Am}{r^2 - b^2}, \quad z = \frac{An}{r^2 - c^2},$$

$$\text{and} \quad \frac{l^2}{r^2 - a^2} + \frac{m^2}{r^2 - b^2} + \frac{n^2}{r^2 - c^2} = 0.$$

Let r_1 and r_2 be the two roots of this equation;

then
$$\frac{l^2}{r_1^2 - a^2} + \frac{m^2}{r_1^2 - b^2} + \frac{n^2}{r_1^2 - c^2} = 0,$$

$$\frac{l^2}{r_2^2 - a^2} + \frac{m^2}{r_2^2 - b^2} + \frac{n^2}{r_2^2 - c^2} = 0.$$

Subtracting,

$$\frac{l^2(r_1^2 - r_2^2)}{(r_1^2 - a^2)(r_2^2 - a^2)} + \frac{m^2(r_1^2 - r_2^2)}{(r_1^2 - b^2)(r_2^2 - b^2)} + \frac{n^2(r_1^2 - r_2^2)}{(r_1^2 - c^2)(r_2^2 - c^2)} = 0;$$

dividing by $r_1^2 - r_2^2$, and multiplying by A^2,

$$x_1x_2 + y_1y_2 + z_1z_2 = 0;$$

which, as $x_1y_1z_1$, $x_2y_2z_2$, are proportional to the direction cosines of the lines, shows that they are at right angles to each other.

4. *Problem in the Papers of* 1842. A quadrilateral, composed of four unequal beams jointed together at the extremities, is compressed by a given force in the direction of one diagonal: find the force in the direction of the other diagonal which will resist the compression.

Let the quadrilateral be $ABCD$, and let the diagonals AC, BD, intersect in O. Then if P be the compressing force in AC, and Q the resisting force in BD, and X be the force in AD, we have, considering AB as a lever turning round B as a fulcrum, and acted on by forces P and X at A,

$$P \sin BAC = X \sin BAD.$$

In the same way, considering CD as a lever turning on a fulcrum at C, and acted on by Q and X at D, we have

$$Q \sin BDC = X \sin CDA;$$

therefore $$\frac{P}{Q} = \frac{\sin BDC}{\sin BAC} \cdot \frac{\sin BAD}{\sin CDA}.$$

But in the triangle ABD we have

$$\frac{\sin BAD}{\sin ABD} = \frac{BD}{AD};$$

in the triangle ACD, $$\frac{\sin CDA}{\sin ACD} = \frac{AC}{AD};$$

therefore $$\frac{P}{Q} = \frac{BD}{AC} \frac{\sin ABD}{\sin ACD} \cdot \frac{\sin BDC}{\sin BAC}.$$

In the triangle ABO, $$\frac{\sin ABD}{\sin BAC} = \frac{AO}{BO};$$

in the triangle CDO, $$\frac{\sin BDC}{\sin ACD} = \frac{CO}{DO};$$

hence $$\frac{P}{Q} = \frac{BD}{AC} \cdot \frac{AO . CO}{BO . OD}.$$

If the quadrilateral be a parallelogram, $AO = OC$ and $BO = OD$, so that $$\frac{P}{Q} = \frac{AC}{BD},$$

or the forces are directly as the diagonals.

γ.

END OF VOL. III.

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